We have $\mathcal{A} = ig \langle \vec{p}, \vec{q} | \int d^4x \, \phi(x) \chi^+(x) \chi(x) | \vec{r} \rangle$ ⊛,

with

$$\phi(x) = \sum_{\vec{p}} \frac{1}{\sqrt{2V\omega_{\vec{p}}}} \left(\hat{c}_{\vec{p}} e^{-ipx} + \hat{c}^+_{\vec{p}} e^{ipx} \right),$$

$$\chi(x) = \sum_{\vec{p}} \frac{1}{\sqrt{2V\omega_{\vec{p}}}} \left(\hat{a}_{\vec{p}} e^{-ipx} + \hat{b}^+_{\vec{p}} e^{ipx} \right),$$

$$\chi^+(x) = \sum_{\vec{p}} \frac{1}{\sqrt{2V\omega_{\vec{p}}}} \left(\hat{b}_{\vec{p}} e^{-ipx} + \hat{a}^+_{\vec{p}} e^{ipx} \right),$$

and

$$|\vec{r}\rangle = \hat{c}^+_{\vec{r}} |0\rangle, \quad |\vec{p}, \vec{q}\rangle = \hat{a}^+_{\vec{p}} \hat{b}^+_{\vec{q}} |0\rangle .$$

Plugging the above into ⊛, we see that the only nonvanishing term is the one with equal number of creation & annihilation operators:

$$\mathcal{A} = ig \sum_{\vec{p}', \vec{q}', \vec{r}'} \frac{1}{\sqrt{V^3 2\omega_{\vec{p}'} 2\omega_{\vec{q}'} 2\omega_{\vec{r}'}}} \int d^4x \, \langle 0 | \hat{b}_{\vec{q}} \hat{a}_{\vec{p}} \hat{c}_{\vec{r}'} \hat{a}^+_{\vec{q}'} \hat{b}^+_{\vec{p}'} \hat{c}^+_{\vec{r}} | 0 \rangle e^{i(p'+q'-r')x}$$

$$= \frac{ig}{\sqrt{V^3 2\omega_{\vec{p}} 2\omega_{\vec{q}} 2\omega_{\vec{r}}}} (2\pi)^4 \delta^{(4)}(p+q-r) .$$

(2) From the above result & the hint, we find:

$$\frac{|\mathcal{A}|^2}{T} = \frac{g^2}{V^2 2\omega_{\vec{p}} 2\omega_{\vec{q}} 2\omega_{\vec{r}}} (2\pi)^4 \delta^{(4)}(p+q-r)$$

Therefore the differential decay rate is

$$d\Gamma = \frac{g^2}{(2\pi)^2\, 2\omega_{\vec{p}}\, 2\omega_{\vec{q}}\, 2\omega_{\vec{r}}}\, d^3\vec{p}\, d^3\vec{q}\, \delta^{(4)}(p+q-r)$$

Yukawa Theory

$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi + \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{1}{2}M^2\varphi^2 - g\varphi\bar{\psi}\psi$

φ real scalar field

$g > 0$

$\boxed{A}$ (1) $\varphi \to \psi + \bar{\psi}$, $M > 2m$

$i(2\pi)^4\delta^{(4)}(p_1+p_2-k)\mathcal{M} = -ig\langle f|\int d^4x\, \bar{\psi}\psi\varphi|i\rangle$

$|i\rangle = |\vec{k}\rangle = c^\dagger(\vec{k})|0\rangle$; $c^\dagger(\vec{k})$: creation operator associated w/ φ

$|f\rangle = |\vec{p}_1, s_1; \vec{p}_2, s_2\rangle = a^\dagger_{s_1}(\vec{p}_1)\, b^\dagger_{s_2}(\vec{p}_2)|0\rangle$

$$\varphi(x) = \int\frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}}\left[c(\vec{p})e^{-ip\cdot x} + c^\dagger(\vec{p})e^{ip\cdot x}\right]$$

$$\psi(x) = \int\frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}}\sum_{i=s_1,s_2}\left[u_i(\vec{p})\,a_i(\vec{p})\,e^{-ip\cdot x} + v_i(\vec{p})\,b_i^\dagger(\vec{p})\,e^{ip\cdot x}\right]$$

$$\bar{\psi}(x) = \int\frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}}\sum_{i=s_1,s_2}\left[\bar{u}_i(\vec{p})\,a_i^\dagger(\vec{p})e^{ip\cdot x} + \bar{v}_i(\vec{p})\,b_i(\vec{p})\,e^{-ip\cdot x}\right]$$

Only one term contributes; the one with equal number of creation & annihilation operators:

$\langle f|\int d^4x\,\bar{\psi}\psi\varphi|i\rangle = \int d^4x\, e^{i(-k+p_1+p_2)\cdot x}\,\bar{u}_{s_1}(\vec{p}_1)v_{s_2}(\vec{p}_2) = (2\pi)^4\delta^{(4)}(p_1+p_2-k)\,\bar{u}_{s_1}(\vec{p}_1)v_{s_2}(\vec{p}_2)$

$\Rightarrow \mathcal{M} = -g\,\bar{u}_{s_1}(\vec{p}_1)v_{s_2}(\vec{p}_2)$

(2) $\sum_s|\mathcal{M}|^2 = g^2\sum_{s_1,s_2}\bar{u}_{s_1}(\vec{p}_1)v_{s_2}(\vec{p}_2)\bar{v}_{s_2}(\vec{p}_2)u_{s_1}(\vec{p}_1) \overset{\text{PS8, ex.3}}{=} g^2\sum_{s_1}\bar{u}_{s_1}(\vec{p}_1)(p\!\!\!/_2 - m)u_{s_1}(\vec{p}_1)$

this is a trace; use cyclic property of trace

$\sum_s|\mathcal{M}|^2 = g^2\,\mathrm{tr}\left[\sum_{s_1}u_{s_1}(\vec{p}_1)\bar{u}_{s_1}(\vec{p}_1)(p\!\!\!/_2 - m)\right] \underset{\text{PS8, ex.3}}{=} g^2\,\mathrm{tr}\left[(p\!\!\!/_1+m)(p\!\!\!/_2-m)\right] =$

$= 4g^2(p_1\cdot p_2 - m^2)$

$\mathrm{tr}[\gamma^\mu] = 0$
$\mathrm{tr}[\gamma^\mu\gamma^\nu] = 4\eta^{\mu\nu}$

center of mass frame, en. & mom. conservation

$s = M^2 = (p_1+p_2)^2 = p_1^2 + p_2^2 + 2p_1\cdot p_2 = 2m^2 + 2p_1\cdot p_2 \Leftrightarrow p_1\cdot p_2 = \frac{M^2}{2} - m^2$

$\Rightarrow \sum_s|\mathcal{M}|^2 = 2g^2(M^2 - 4m^2)$

$\delta^{(4)}(p_1+p_2-k) = \delta(\omega_{\vec{p}_1} + \omega_{\vec{p}_2} - M)\,\delta^{(3)}(\vec{p}_1+\vec{p}_2)$

$\omega_{\vec{k}} = M$; $\omega_{\vec{p}_i}^2 = m^2 + \vec{p}_i^{\,2}$; $i = 1,2$

$\frac{d\omega}{dp} = \frac{1}{2\omega}2p \Rightarrow p\,dp = \omega\,d\omega$ $\qquad p \equiv |\vec{p}|$

$$\Gamma = \frac{1}{2M} 2g^2 (M^2 - 4m^2) \int \frac{d^3\vec{p}_1}{(2\pi)^3 2\omega_{\vec{p}_1}} \frac{(2\pi)}{2\omega_{\vec{p}_1}} \underbrace{\delta(2\omega_{\vec{p}_1} - M)}_{= \frac{1}{2}\delta(\omega_{\vec{p}_1} - \frac{M}{2})} =$$

$$= g^2 M \left(1 - \frac{4m^2}{M^2}\right) \int \frac{\overbrace{dp_1 \cdot p_1}^{d\omega_{\vec{p}_1} \cdot \omega_{\vec{p}_1}} p_1 \overbrace{d\Omega}^{4\pi}}{(2\pi)^2 \; 8\omega_{\vec{p}_1}^2} \delta\left(\omega_{\vec{p}_1} - \frac{M}{2}\right) =$$

$$p_1 = (\omega_{\vec{p}_1}^2 - m^2)^{1/2}$$

$$= g^2 M \left(1 - \frac{4m^2}{M^2}\right) \frac{\frac{M}{2}\left(\frac{M^2}{4} - m^2\right)^{1/2} 4\pi}{4\pi^2 \, 2M^2} = \frac{g^2 M}{8\pi}\left(1 - \frac{4m^2}{M^2}\right)^{3/2}$$

(3) *Lifetime* $\tau = \Gamma^{-1}$

$m_e = 5 \cdot 10^{-4}\,\mathrm{GeV}$; $M = 125\,\mathrm{GeV}$; $g \simeq \frac{m_e}{M} \approx 4 \cdot 10^{-6} \ll 1$

$$\Rightarrow \tau \simeq \left.\frac{8\pi}{g^2 M}\right|_{g = \frac{m}{M}} = \frac{8\pi M}{m_e^2} \sim 10^{10}\,\mathrm{GeV}^{-1} \sim 10^{-14}\,s \approx 10^8 \tau_{Higgs}$$

$m_\tau = 1{,}7\,\mathrm{GeV}$; $g \ll 1$

$$\Rightarrow \tau \simeq \frac{8\pi M}{m_\tau^2} \sim 10^3\,\mathrm{GeV}^{-1} \sim 10^{-21}\,s \approx \tau_{Higgs}$$

Scattering cross section

① Flux = number density ⊗ velocity of particle.

Since we have one particle per box,

Flux $= \frac{1}{V} \cdot |\vec{v}|$, with $|\vec{v}|$ the velocity of the moving particle.

② We start from $p_1^\mu \cdot p_{2\mu} = \omega_1 \omega_2 (1 - v_1 v_2)$

$$\rightarrow (p_1^\mu \cdot p_{2\mu})^2 = \omega_1^2 \omega_2^2 (1 - v_1 v_2)^2 ,$$

therefore:

$$(p_1^\mu \cdot p_{2\mu})^2 - m_1^2 m_2^2 = (p_1^\mu \cdot p_{2\mu}) - (p_1^\mu \cdot p_{1\mu})(p_2^\mu \cdot p_{2\mu})$$

$$= \omega_1^2 \omega_2^2 \left[(1 - v_1 v_2)^2 - (1 - v_1^2)(1 - v_2^2) \right]$$

$$= \dots = \omega_1^2 \omega_2^2 (v_1 - v_2)^2 ,$$

where we used $p_i^\mu p_{i\mu} = m_i^2$, $i = 1, 2$.

(3) As in the previous exercise, one can argue that for the process $p_1 + p_2 \to k_1 + k_2$,

$$d\sigma = \frac{1}{|\vec{v}_1 - \vec{v}_2| 2\omega_{\vec{p}_1} 2\omega_{\vec{p}_2}} \frac{d^3\vec{k}_1}{(2\pi)^3 2\omega_{\vec{k}_1}} \frac{d^3\vec{k}_2}{(2\pi)^3 2\omega_{\vec{k}_2}} (2\pi)^4 \delta^{(4)}(p_1+p_2-k_1-k_2) |\mathcal{M}|^2$$

(4) We are asked to compute the phase-space integral

$$I = \int \frac{d^3\vec{k}_1}{(2\pi)^3 2\omega_{\vec{k}_1}} \int \frac{d^3\vec{k}_2}{(2\pi)^3 2\omega_{\vec{k}_2}} (2\pi)^4 \delta^{(4)}(p - k_1 - k_2),$$

with $p \equiv p_1 + p_2$. We know that the above integral is Lorentz-invariant, so we can compute it in any frame we want. Thus, we pick the most convenient one, which is the center of mass frame: $\vec{p} = 0$, $p^0 = \omega_{\vec{p}_1} + \omega_{\vec{p}_2} \equiv E$

$$\Rightarrow I = \int \frac{d^3\vec{k}_1}{(2\pi)^3 2\omega_{\vec{k}_1}} \int \frac{d^3\vec{k}_2}{(2\pi)^3 2\omega_{\vec{k}_2}} (2\pi)^4 \delta(E - \sqrt{k_1^2+m_1^2} - \sqrt{k_2^2+m_2^2}) \delta^{(3)}(\vec{k}_1 + \vec{k}_2)$$

$$= \frac{1}{4(2\pi)^2} \int \frac{dk_1 \, k_1^2}{\sqrt{k_1^2+m_1^2}\sqrt{k_1^2+m_2^2}} \delta(E - \sqrt{k_1^2+m_1^2} - \sqrt{k_1^2+m_2^2}) \int d\Omega$$

Since $\int dx \, \delta[f(x)] \, s(x) = \sum_i \frac{s(x_i)}{|f'(x_i)|}$, with x_i = zeros of $f(x)$, We find:

$$I = \frac{k_1}{16\pi^2 E} \int d\Omega ,$$

where k_1 is the solution of

$$E - \sqrt{\tilde{k}_1^2 + m_1^2} - \sqrt{\tilde{k}_1^2 + m_2^2} = 0 ,$$

meaning that

$$k_1 = \frac{1}{2E} \sqrt{[E^2 - (m_1+m_2)^2][E^2 - (m_1-m_2)^2]}$$

⑤ Using the previous results,

$$\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{|\vec{v}_1 - \vec{v}_2| \, 2\omega_{\vec{p}_1} \, 2\omega_{\vec{p}_2}} \frac{k_1}{16\pi^2 E}$$

In the center of mass frame,

$$|\vec{v}_1 - \vec{v}_2| = \left| \frac{\vec{p}_1}{\omega_{\vec{p}_1}} - \frac{\vec{p}_2}{\omega_{\vec{p}_2}} \right| = \frac{1}{\omega_{\vec{p}_1}\omega_{\vec{p}_2}} |\omega_{\vec{p}_2} \vec{p}_1 - \omega_{\vec{p}_1} \vec{p}_2|$$

$$= \frac{p_1 E}{\omega_{\vec{p}_1}\omega_{\vec{p}_2}} , \quad \text{since } \vec{p}_1 + \vec{p}_2 = 0 \ \& \ \omega_{\vec{p}_1} + \omega_{\vec{p}_2} = E .$$

$$\rightarrow \frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{64\pi^2 E^2} = \frac{|\mathcal{M}|^2}{64\pi^2 E^2} \frac{k_1}{p_1} = \frac{\lambda^2}{64\pi^2 E^2} ,$$

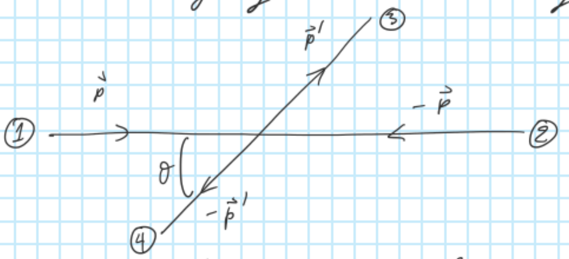
since for $\lambda \varphi^4$, $|\mathcal{M}|^2 = \lambda^2$ & $m_1 = m_2 = m \Rightarrow k_1 = p_1$.

The total cross section is

$$\sigma = \frac{1}{2}\int d\Omega \frac{d\sigma}{d\Omega},$$ where the $\frac{1}{2}$ factor was introduced to account for the fact that the particles are indisguishable.

Mandelstam variables

We are studying the scattering process



We know that $p_i^2 = m_i^2$, $i = 1, 2, 3, 4$ &

$p_1 + p_2 = p_3 + p_4$ (conservation of energy & momentum).

$$s = (p_1+p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2 = m_1^2 + m_2^2 + 2\, p_1 \cdot p_2 .$$

$$\rightsquigarrow \quad s = m_1^2 + m_2^2 + 2E_1E_2 - 2\vec{p}_1 \cdot \vec{p}_2 \qquad \circledast$$

At the center of mass frame,

$\vec{p}_1 + \vec{p}_2 = 0 \rightarrow \vec{p}_1 = -\vec{p}_2 = \vec{p}$

$\Rightarrow p_1 = (E_1, \vec{p})$, $p_2 = (E_2, -\vec{p})$

meaning that $p_1 + p_2 = (E_1 + E_2, \vec{0})$

$\Rightarrow E_2 = \sqrt{s} - E_1$

Plugging this into $\circledast$, we find

$$\boxed{E_1 = \frac{1}{2\sqrt{s}}(s + m_1^2 - m_2^2)}$$

Similarly, we find that $\boxed{E_2 = \frac{1}{2\sqrt{s}}(s + m_2^2 - m_1^2)}$,

$$\boxed{E_3 = \frac{1}{2\sqrt{s}}(s + m_3^2 - m_4^2)} \quad , \quad \boxed{E_4 = \frac{1}{2\sqrt{s}}(s + m_4^2 - m_3^2)}$$

For the three-momenta, we find

$$|\vec{p}|^2 = E_1^2 - m_1^2 = \frac{1}{4s}\left[(s + m_1^2 - m_2^2)^2 - 4s\,m_1^2\right]$$

$$\rightarrow \boxed{|\vec{p}|^2 = \frac{1}{4s}\lambda(s, m_1^2, m_2^2)} \; ,$$

where $\lambda(x, y, z)$ is the "Källen function"

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz \, .$$

Also $\boxed{|\vec{p}\,'|^2 = E_3^2 - m_3^2 = \frac{1}{4s}\lambda(s, m_3^2, m_4^2)}$.

For the scattering angle, let's first look at t

$$t = (p_1 - p_3)^2 = m_1^2 + m_3^2 - 2E_1E_3 + 2|\vec{p}||\vec{p}\,'|\cos\vartheta \, .$$

$$u = (p_1 - p_4)^2 = m_1^2 + m_4^2 - 2E_1E_4 - 2|\vec{p}||\vec{p}\,'|\cos\vartheta$$

$$t - u = m_3^2 - m_4^2 - 2E_1(E_3 - E_4) + 4|\vec{p}||\vec{p}\,'|\cos\vartheta$$

$$\rightarrow \cos\vartheta = \frac{1}{4|\vec{p}||\vec{p}\,'|}\left(t - u - m_3^2 + m_4^2 + 2E_1(E_3 - E_4)\right)$$

$$\boxed{\cos\vartheta = \frac{s(t-u) + (m_1^2 - m_2^2)(m_3^2 - m_4^2)}{\sqrt{\lambda(s, m_1^2, m_2^2)\,\lambda(s, m_3^2, m_4^2)}}}$$

$$s + t + u = (p_1 + p_2)^2 + (p_1 - p_3)^2 + (p_1 - p_4)^2$$

$$= m_1^2 + m_2^2 + m_3^2 + m_4^2 + 2p_1(p_1 + p_2 - p_3 - p_4)$$