Quantum Field Theory (Quantum Electrodynamics)

Problem Set 10 8 & 10 January 2024

1. Decay rates

Part A.

Consider the following theory describing the (cubic) interaction of a real field ϕ with the complex scalar χ

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^2 - \frac{1}{2}m_\phi^2\phi^2 + |\partial_\mu\chi|^2 - m_\chi^2|\chi|^2 - g\phi|\chi|^2 \ ,$$

where g is a real parameter with dimension 1. The decomposition of ϕ in the box normalization reads

$$\phi(x) = \sum_{\vec{p}} \frac{1}{\sqrt{2V\omega_{\vec{p}}}}\left(\hat{c}_{\vec{p}}e^{-ipx} + \hat{c}_{\vec{p}}^+ e^{ipx}\right),$$

while $\chi(x)$ is decomposed as in part A.

1. Let us consider the decay process $\phi \to \chi + \chi^+$. At the lowest order in g, the transition amplitude between the initial state $|\vec{r}\rangle = \hat{c}_{\vec{r}}^+|0\rangle$ (describing the "incoming" particle ϕ with momentum $\vec{r}$) and the final state $|\vec{p},\vec{q}\rangle = \hat{a}_{\vec{p}}^+\hat{b}_{\vec{q}}^+|0\rangle$ (describing the "outgoing" particles χ and χ^+ with momenta $\vec{p}$ and $\vec{q}$, respectively) reads

$$i(2\pi)^4\delta^{(4)}(p_1+p_2-k)\mathcal{M} = ig\,\langle\vec{p},\vec{q}|\int \mathrm{d}^4x\,\phi(x)\chi^+(x)\chi(x)\,|\vec{r}\rangle\,.$$

Show that

$$\mathcal{M} = \frac{g}{\sqrt{V^3 2\omega_{\vec{p}}2\omega_{\vec{q}}2\omega_{\vec{r}}}}\,.$$

2. Now we are in position to calculate the differential decay rate for the ϕ-particle to decay into χ and χ^+ with momenta in the interval $(\vec{p}+\mathrm{d}\vec{p})$ and $(\vec{q}+\mathrm{d}\vec{q})$. We translate back to the infinite-space-normalization, making use of the fact that the interval $\mathrm{d}^3\vec{p}\,\mathrm{d}^3\vec{q}$ contains the following number of states :

$$\frac{V\mathrm{d}^3\vec{p}}{(2\pi)^3}\frac{V\mathrm{d}^3\vec{q}}{(2\pi)^3}\,.$$

The transition probability per unit time is $\frac{|\mathcal{A}|^2}{T}$, where $T = \int dt$. Show that the differential decay rate is given by

$$\mathrm{d}\Gamma = \frac{g^2}{32\pi^2\prod_i\omega_i}\mathrm{d}^3\vec{p}\,\mathrm{d}^3\vec{q}\,\delta^{(4)}(p+q-r),$$

where the ω_i's are the frequencies of the external particles.

Useful formula : $\left[(2\pi)^4\delta^{(4)}(p+q-r)\right]^2 = (2\pi)^4\delta^{(4)}(p+q-r)VT\,.$

Part B.

Let us next consider the following Lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}M^2\phi^2 - g\phi\bar{\psi}\psi$$

where ϕ is a real scalar field, $g > 0$ and $m,\ M$ are the fermionic and scalar masses, respectively.

(a) Let us consider the decay process $\phi \to \psi + \bar{\psi}$, which can take place if $M > 2m$. At the lowest order in g, the transition amplitude $\mathcal{M}$ between the states $|i\rangle$ and $|f\rangle$ is defined as

$$i(2\pi)^4\delta^{(4)}(p_1 + p_2 - k)\mathcal{M} = -ig\,\langle f| \int \mathrm{d}^4x\, \bar{\psi}\psi\phi\,|i\rangle \ .$$

Compute the above for an initial state corresponding to one ϕ particle with momentum $\vec{k}$, and a final state comprising a fermion and an antifermion with momenta $\vec{p}_1$ and $\vec{p}_2$ and spins s_1 and s_2, respectively.

(b) Working in the center-of-mass frame, compute the decay rate

$$\Gamma = \frac{1}{2\omega_{\vec{k}}}\sum_{s_1,s_2}\int \frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^3 2\omega_{\vec{p}_1}}\frac{\mathrm{d}^3\vec{p}_2}{(2\pi)^3 2\omega_{\vec{p}_2}}(2\pi)^4\delta^{(4)}(p_1 + p_2 - k)|\mathcal{M}|^2 \ ,$$

where the summation over spins takes care of the fact that the decay includes all possible final states for the fermion–antifermion pair.

Hint : Using the formulas from previous Problem Sets, show that

$$\sum_{s_1,s_2}|\mathcal{M}|^2 = 2g^2(M^2 - 4m^2) \ .$$

(c) For $m = 0.5$ MeV (electron), $M = 125$ GeV (Higgs particle), and $g \approx m/M$, compute the lifetime of the scalar in seconds. Compare this to the measured lifetime of the Higgs particle, $\tau = 1.5 \cdot 10^{-22}$ s. Estimate the lifetime for decay into tau leptons, whose mass is $m = 1.7$ GeV.

3. Scattering cross section

The purpose of this exercise is to study the scattering between two particles. In the laboratory, one would typically use two beams of particles (or a beam on a target), so the interaction rate would depend on the relative velocities and the particle number densities. In order to obtain a quantity which is independent of these parameters, one defines the differential cross section as the differential interaction rate divided by the flux of incident particles.

1. In the box normalization, there is one particle per volume. Argue why in the rest frame of one of the particles the flux is given by v/V, where $v = |\vec{v}|$ is the magnitude of the velocity of the other particle.
2. Show that in a collinear frame, where $p_1^\mu = \omega_{\vec{p}_1}(1,0,0,v_1)$ and $p_2^\mu = \omega_{\vec{p}_2}(1,0,0,v_2)$,

$$(p_1^\mu p_{2\mu})^2 - m_1^2 m_2^2 = (\omega_{\vec{p}_1}\omega_{\vec{p}_2}(v_1 - v_2))^2 \ .$$

3. Use the previous results to show that the differential cross section is given by

$$\mathrm{d}\sigma = \frac{1}{|\vec{v}_1 - \vec{v}_2| 2\omega_{\vec{p}_1} 2\omega_{\vec{p}_2}} \frac{\mathrm{d}^3 k_1}{(2\pi)^3 2\omega_{\vec{k}_1}} \frac{\mathrm{d}^3 k_2}{(2\pi)^3 2\omega_{\vec{k}_2}} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - k_1 - k_2) |\mathcal{M}|^2,$$

where $\mathcal{M}$ is the so-called invariant amplitude, containing just the parts of the process which do not depend on kinematics.

4. Calculate the phase space integral

$$\int \frac{\mathrm{d}^3 \vec{k}_1}{(2\pi)^3 2\omega_{\vec{k}_1}} \frac{\mathrm{d}^3 \vec{k}_2}{(2\pi)^3 2\omega_{\vec{k}_2}} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - k_1 - k_2).$$

5. Calculate the total cross section at first order in the coupling constant λ for $2 \to 2$ scattering in the ϕ^4 theory.

   Hint : Note that the two particles in the final state are indistinguishable.

4. Mandelstam variables

Let us consider a $2 \to 2$ scattering process. The particles in the initial state have masses m_1 and m_2 and four-momenta p_1 and p_2, while the ones in the final state have masses m_3 and m_4 and four-momenta p_3 and p_4. The *Mandelstam variables* are defined as

$$s = (p_1 + p_2)^2 \ , \quad t = (p_1 - p_3)^2 \ , \quad u = (p_1 - p_4)^2 \ .$$

1. Express the energies E_i, the three-momenta $|\vec{p}_i|$ and the scattering angle θ in the center-of-mass frame in terms of s, t, u and the particle masses.
2. Show that

$$s + t + u = \sum_{i=1}^{4} m_i^2 \ .$$