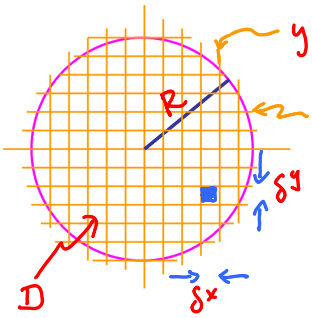
C4 Integration in krummlinigen Koordinaten

Falls ein System Symmetrien hat (z.B. invariant unter Rotationen um eine Symmetrie-Achse), lassen sich Integrale durch Nutzung krummliniger Koordinaten einfacher berechnen.

Beispiel: Kreisfläche



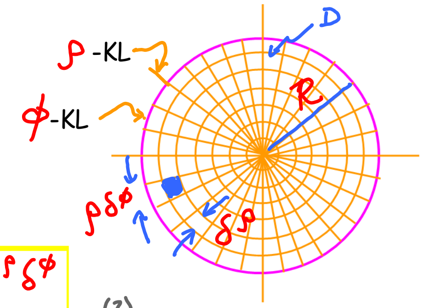
'D' für 'disk'

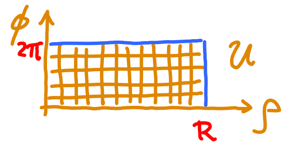
$$\delta S = \delta^x \delta^y \quad (1)$$

Flächen-element

$$\boxed{\delta S = \rho\, \delta^\rho \delta^\phi} \quad (2)$$

'S' für 'surface'





Nach geeigneter Koordinatentransformation wird aus der Scheibe

$$D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le R^2\}$$ ein 'Rechteck': $$U = \left\{ (\rho,\phi) \in \mathbb{R}^2 : \begin{array}{l} \rho \in (0,R); \\ \phi \in (0,2\pi) \end{array} \right\} \quad (3)$$

Kreis-fläche:

$$A = \int_S \underbrace{dx\,dy}_{\text{Kartesisches Flächenelement (1)}} \cdot 1 \overset{(2)}{=} \int_U \underbrace{\rho\, d\phi\, d\rho}_{\text{Polares Flächenelement (2)}} = \int_0^R d\rho \cdot \rho \int_0^{2\pi} d\phi = \frac{1}{2} R^2 \cdot 2\pi = \pi R^2 \quad (4)$$

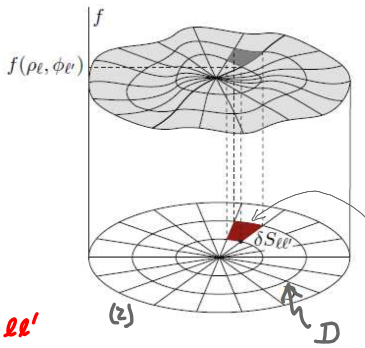
'A' für 'area'

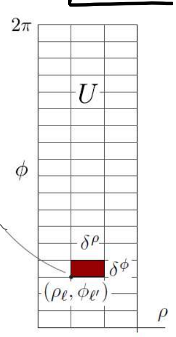
[eleganter als auf Seite C4e !]

C4.2 2D-Integral in Polarkoordinaten

$$\int_D dS\, f(\vec{r}) \simeq \sum_{\ell\ell'} |\delta S_{\ell\ell'}|\, f(\rho_\ell, \phi_{\ell'}) \quad (1)$$

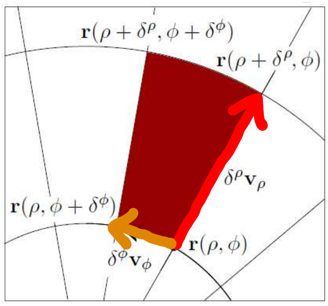
Flächenelement $\delta S_{\ell\ell'}$ wird aufgespannt durch:





$$\vec{r}(\rho_\ell + \delta^\rho, \phi_{\ell'}) - \vec{r}(\rho_\ell, \phi_{\ell'}) = \delta^\rho \partial_\rho \vec{r}(\rho_\ell, \phi_{\ell'}) \overset{(V2e.2)}{=} \delta^\rho (\vec{v}_\rho)_{\ell\ell'}$$

$$\vec{r}(\rho_\ell, \phi_{\ell'} + \delta^\phi) - \vec{r}(\rho_\ell, \phi_{\ell'}) = \delta^\phi \partial_\phi \vec{r}(\rho_\ell, \phi_{\ell'}) \overset{(V2e.4)}{=} \delta^\phi (\vec{v}_\phi)_{\ell\ell'} \quad (3)$$



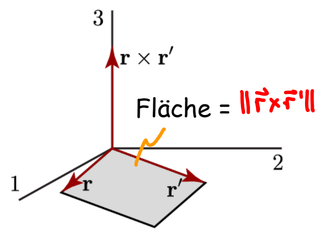
Geometrische Fläche von $\delta S_{\ell\ell'}$:

$$|\delta S_{\ell\ell'}| = \delta^\rho \delta^\phi \, \| (\vec{v}_\rho)_{\ell\ell'} \times (\vec{v}_\phi)_{\ell\ell'} \| \quad (4)$$

$$\int_D dS\, f(\vec{r}) = \int_0^R d\rho \int_0^{2\pi} d\phi\, f(\rho,\phi)\, \underbrace{\| \vec{v}_\rho \times \vec{v}_\phi \|}_{|v_\rho v_\phi|\, \|\vec{e}_\rho \times \vec{e}_\phi\|} \quad (5)$$

(V2I.4): $v_\rho = 1$, $v_\phi = \rho$; $\quad \|\vec{e}_\rho \times \vec{e}_\phi\| = 1$ (V2I.5) $\quad (6)$

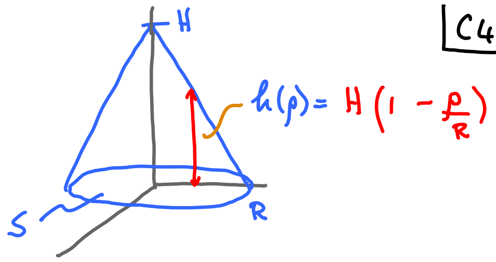
$$= \int_0^R \rho\, d\rho \int_0^{2\pi} d\phi\, f(\rho,\phi) \quad (7)$$



'Integrationsmaß' in Polarkoordinaten: $\boxed{{}'dS = \rho\, d\rho\, d\phi'}$ (8)

C4j

Beispiel: Volumen eines Kegels



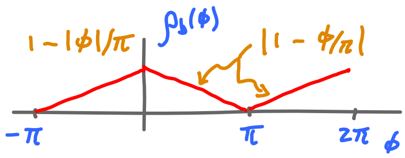
$$V = \int_D dS\; h(\rho) \qquad (1)$$

$$= \int_0^R d\rho \cdot \rho \int_0^{2\pi} d\phi \quad H\,(1 - \rho/R) \qquad (2)$$

$$= 2\pi\, H \int_0^R d\rho\, (\rho - \rho^2/R) = 2\pi\, H \left[\tfrac{1}{2}\rho^2 - \tfrac{1}{3}\rho^3/R\right]_0^R = 2\pi\, H\, \tfrac{1}{6} R^2 = \tfrac{1}{3}\pi H R^2 \qquad (3)$$

Beispiel: Herz-Fläche (geschachtelte Integrationsgrenzen)

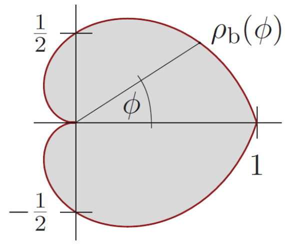
Abstand vom Ursprung zum Rand des Herzens:



$$\rho_b(\phi) = (1 - |\phi|/\pi), \qquad \phi \in (-\pi, \pi) \qquad (4a)$$

oder

$$\rho_b(\phi) = |1 - \phi/\pi|, \qquad \phi \in (0, 2\pi) \qquad (4b)$$

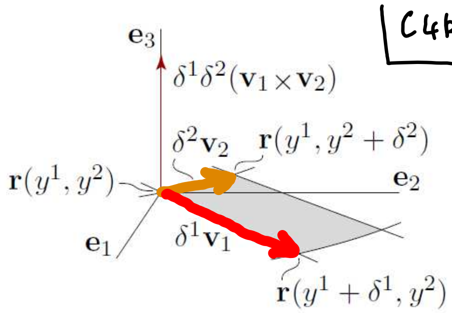


$$\text{Fläche} = \int_{-\pi}^{\pi} d\phi \int_0^{\rho_b(\phi)} d\rho \cdot \rho = \int_{-\pi}^{\pi} d\phi\; \tfrac{1}{2}\underbrace{\rho_b^2(\phi)}_{(1 - |\phi|/\pi)^2} \qquad (5)$$

$$= \tfrac{1}{3}\pi$$

C4k

Allgemeine Koordinatentransformation in 2D



$$\vec{r}: \; \mathcal{U} \subset \mathbb{R}^2 \longrightarrow M \subset \mathbb{R}^2$$

$$\vec{y} = \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} \longmapsto \vec{r}(\vec{y}) = \begin{pmatrix} x^1(y^1, y^2) \\ x^2(y^1, y^2) \end{pmatrix} \qquad (1)$$

Flächenelement wird aufgespannt durch: $\delta^1 \vec{v}_1 = \delta^1 \partial_{y^1} \vec{r}\,, \quad \delta^2 \vec{v}_2 = \delta^2 \partial_{y^2} \vec{r}$ (2)

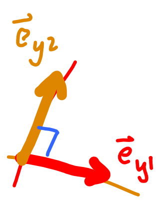
$$|\delta S| \overset{(L4b.1)}{=} \delta^1\delta^2 \, \|\vec{v}_1 \times \vec{v}_2\| = \delta^1\delta^2 \, \|\partial_{y^1}\vec{r} \times \partial_{y^2}\vec{r}\,\| \qquad (3)$$

2D-Integral:

$$\boxed{\int_M dS\, f(\vec{r}) = \int_{\mathcal{U}} dy^1\, dy^2\, \|\partial_{y^1}\vec{r} \times \partial_{y^2}\vec{r}\,\| \; f(\vec{r}(y^1, y^2))} \qquad (4)$$

(V2f.4)

$$\partial_{y^1}\vec{r} \times \partial_{y^2}\vec{r} = \underbrace{\|\vec{v}_{y^1}\|}_{= v_{y^1} = \sqrt{g_{y^1y^1}}} \vec{e}_{y^1} \times \underbrace{\|\vec{v}_{y^2}\|}_{= v_{y^2} = \sqrt{g_{y^2y^2}}} \vec{e}_{y^2} \qquad (5)$$



Für krummlinig-orthogonale Koordinaten gilt:

$$\|\vec{e}_{y^1} \times \vec{e}_{y^2}\| = 1 \qquad (6)$$

Integrationsmaß:

$$\boxed{dS = dy^1\, dy^2 \;\; v_{y^1}\, v_{y^2}} \qquad (7)$$

Bsp: Polarkoord.

$y^1 = \rho \;, \; y^2 = \phi$

$v_{y^1} = 1 \;, \; v_{y^2} = \rho$

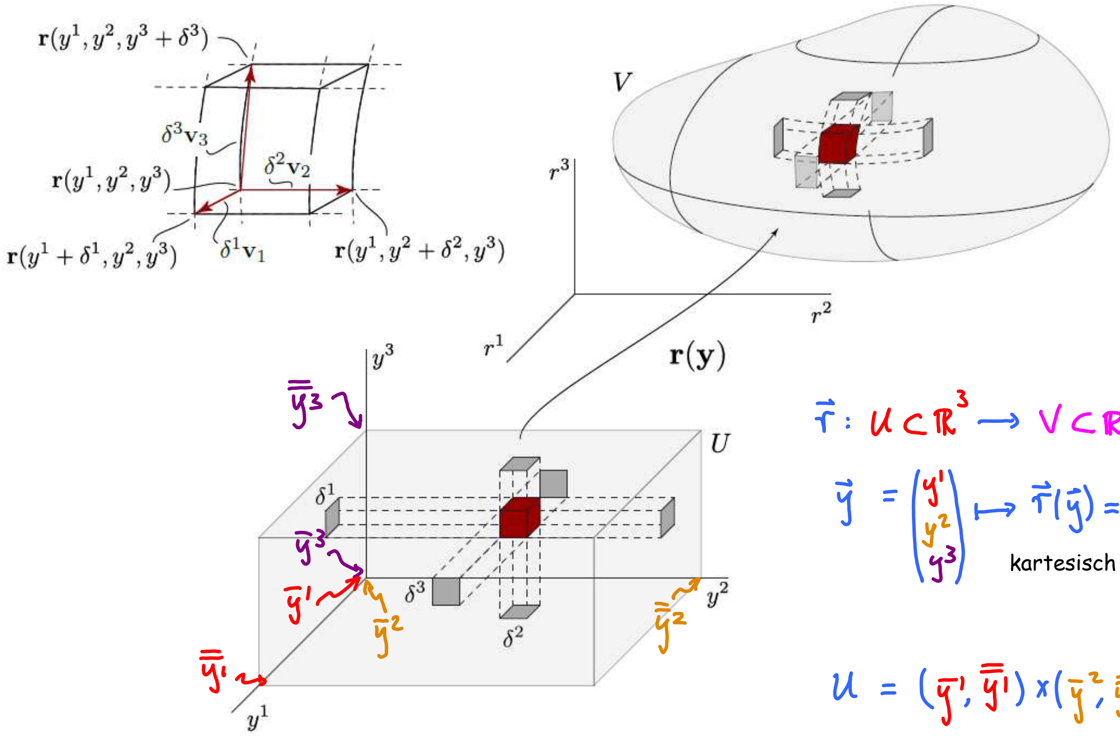
$\vec{e}_{y^1} = \vec{e}_\rho \,, \; \vec{e}_{y^2} = \vec{e}_\phi$

$\vec{e}_{y^1} \times \vec{e}_{y^2} = 1$

$dS \overset{(7)}{=} d\rho\, d\phi \cdot \rho$

C4.3 Volumenintegrale

C4l



$$\vec{r}: \mathcal{U} \subset \mathbb{R}^3 \longrightarrow V \subset \mathbb{R}^3$$

$$\vec{y} = \begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix} \longmapsto \vec{r}(\vec{y}) = \begin{pmatrix} x^1(y^1, y^2, y^3) \\ x^2(y^1, y^2, y^3) \\ x^3(y^1, y^2, y^3) \end{pmatrix}$$

kartesisch

$$\mathcal{U} = (\bar{y}^1, \bar{\bar{y}}^1) \times (\bar{y}^2, \bar{\bar{y}}^2) \times (\bar{y}^3, \bar{\bar{y}}^3)$$

Bsp: Kugelkoordinaten: $y^1 = r, \; y^2 = \theta, \; y^3 = \phi$

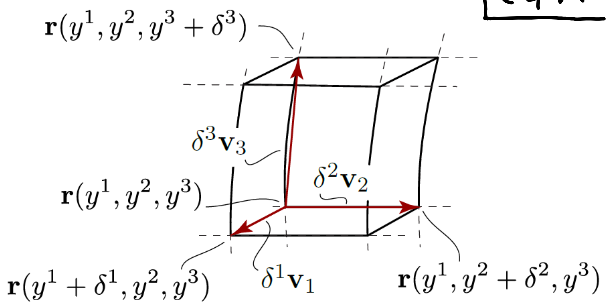
Ball mit Radius R: $\mathcal{U} = (0, R) \times (0, \pi) \times (0, 2\pi)$

Allgemeine Koordinatentransformation in 3D

C4m

$$\vec{r}: \mathcal{U} \subset \mathbb{R}^3 \longrightarrow V \subset \mathbb{R}^3$$

$$\vec{y} \longmapsto \vec{r}(y^1, y^2, y^3) \qquad (1)$$



Volumenelement:

Volumen des Parallelelipeds = Spatprodukt (L4m.4):

$$\delta V = \left| (\vec{v}_1 \times \vec{v}_2) \cdot \vec{v}_3 \;\; \delta^1 \, \delta^2 \, \delta^3 \right| \qquad (2)$$

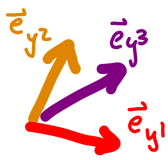
3D-Integral:

$$\boxed{\int_V dV \, f(\vec{r}) = \int_{\bar{y}^1}^{\bar{\bar{y}}^1} dy^1 \int_{\bar{y}^2}^{\bar{\bar{y}}^2} dy^2 \int_{\bar{y}^3}^{\bar{\bar{y}}^3} dy^3 \, \left\| \left(\partial_{y^1} \vec{r} \times \partial_{y^2} \vec{r} \right) \cdot \partial_{y^3} \vec{r} \right\| \, f(\vec{r}(y^1, y^2, y^3))} \qquad (3)$$

(V2f.4): $$\left(\partial_{y^1} \vec{r} \times \partial_{y^2} \vec{r} \right) \cdot \partial_{y^3} \vec{r} = \left(v_{y^1} \vec{e}_{y^1} \times v_{y^2} \vec{e}_{y^2} \right) \cdot \left(v_{y^3} \vec{e}_{y^3} \right) \qquad (4)$$

Für krummlinig-
orthogonale
Koordinaten gilt:

$$\left| \left(\vec{e}_{y^1} \times \vec{e}_{y^2} \right) \cdot \vec{e}_{y^3} \right| = 1 \qquad (5)$$

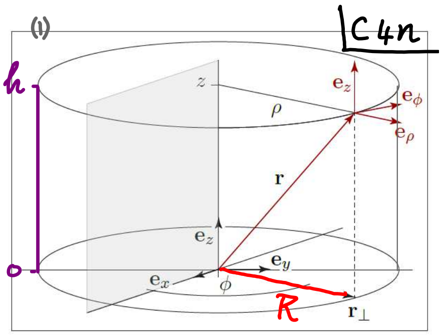


Integrationsmaß:

$$\boxed{dV = dy^1 \, dy^2 \, dy^3 \, v_{y^1} \, v_{y^2} \, v_{y^3}} \qquad (6)$$

Beispiel 1: Zylinderkoordinaten $\quad y^1 = \rho,\ y^2 = \phi,\ y^3 = z$ (1)

C4n



(V2l.1): $\vec r = \vec e_1\, \rho\cos\phi + \vec e_2\, \rho\sin\phi + \vec e_3\, z$ (2)

(V2l.2): $v_\rho = \sqrt{g_{\rho\rho}} = 1,\quad v_\phi = \sqrt{g_{\phi\phi}} = \rho,\quad v_z = \sqrt{g_{zz}} = 1$ (3)

(C4m.6): $\boxed{dV = d\rho\, d\phi\, dz\, \rho}$ (4)

Volumen eines Zylinders: $V = \int_0^R d\rho \cdot \rho \int_0^{2\pi} d\phi \int_0^h dz = \frac{1}{2} R^2 \cdot 2\pi \cdot h = \pi R^2 h$ ✓ (5)

Trägheitsmoment eines homogenen Zylinders: $I \equiv \int_V dV\, \rho(\vec r)\, d_\perp^2(\vec r)$ (6)

Dichte = ρ_0 = Masse/Volumen = $\dfrac{M}{\pi R^2 h}$ (7)

(nicht mit Radius zu verwechseln!)

Abstand v. Symmetrieachse: $d_\perp(\vec r) = \rho$ (8)

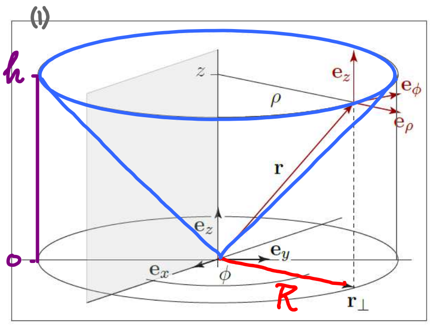
$$I = \int_V dV\, \overbrace{\rho(\vec r)}^{(6):\ \rho_0}\, \overbrace{d_\perp^2(\vec r)}^{(7):\ \rho^2} \overset{(C4l.6)}{=} \int_0^R d\rho \cdot \rho \int_0^{2\pi} d\phi \int_0^h dz\ \rho_0 \cdot \rho^2 = \underbrace{\rho_0}_{(7)\ \frac{M}{\pi R^2 h}} \underbrace{\int_0^R d\rho\, \rho^3}_{\frac{1}{4}R^4} \underbrace{\int_0^{2\pi} d\phi \cdot 1}_{2\pi} \underbrace{\int_0^h dz \cdot 1}_{h} \quad (9)$$

$$= \frac{1}{2} M R^2 \quad (10)$$

Beispiel 2: Volumen eines Kegels

C4o

Zylinderkoordinaten: $y^1 = \rho,\ y^2 = \phi,\ y^3 = z$ (1)



(V2l.1): $\vec r = \vec e_1\, \rho\cos\phi + \vec e_2\, \rho\sin\phi + \vec e_3\, z$ (2)

(V2l.3): $v_\rho = \sqrt{g_{\rho\rho}} = 1,\quad v_\phi = \sqrt{g_{\phi\phi}} = \rho,\quad v_z = \sqrt{g_{zz}} = 1$ (3)

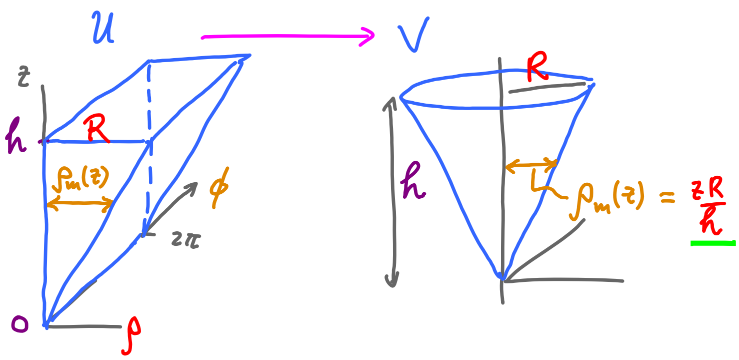
(C4m.6): $\boxed{dV = d\rho\, d\phi\, dz \cdot \rho}$ (4)

$$V = \int_0^h dz \int_0^{\rho_m(z)} d\rho \cdot \rho \int_0^{2\pi} d\phi$$

$$= \int_0^h dz \left[\frac{1}{2}\rho^2\right]_0^{zR/h} 2\pi$$

$$= \pi \left(\frac{R}{h}\right)^2 \underbrace{\int_0^h dz\ z^2}_{\frac{1}{3}h^3}$$

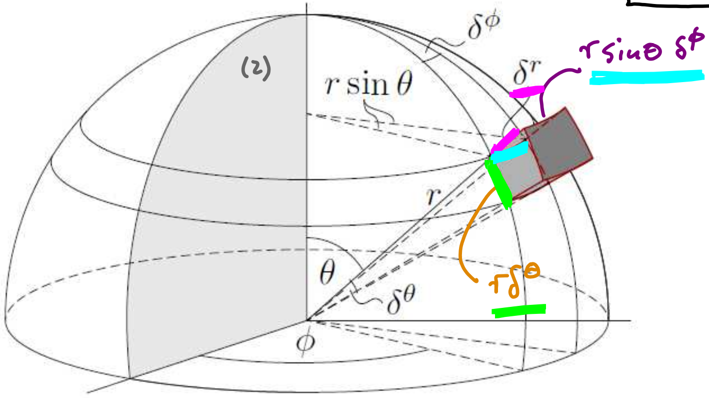
$$= \frac{\pi}{3} h R^2$$



Beispiel 3: Kugelkoordinaten

C4p

$$y^1 = r \,, \quad y^2 = \theta \,, \quad y^3 = \phi \qquad (1)$$



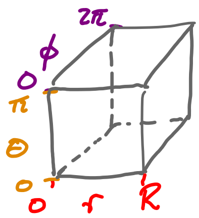
(V2n.1): $$\vec r = \vec e_1 \, r \sin\theta \cos\phi + \vec e_2 \, r \sin\theta \sin\phi + \vec e_3 \, r \cos\theta \qquad (2)$$

(V2n.4: $$v_r = \sqrt{g_{rr}} = 1 \,, \quad v_\theta = \sqrt{g_{\theta\theta}} = r \,, \quad v_\phi = \sqrt{g_{\phi\phi}} = r \sin\theta \qquad (3)$$

Integrationsmass: $$dV \overset{\text{(C4m.6)}}{=} dr \, d\theta \, d\phi \cdot 1 \cdot r \cdot r \sin\theta = \boxed{dr \, d\theta \, d\phi \, r^2 \sin\theta} \qquad (4)$$

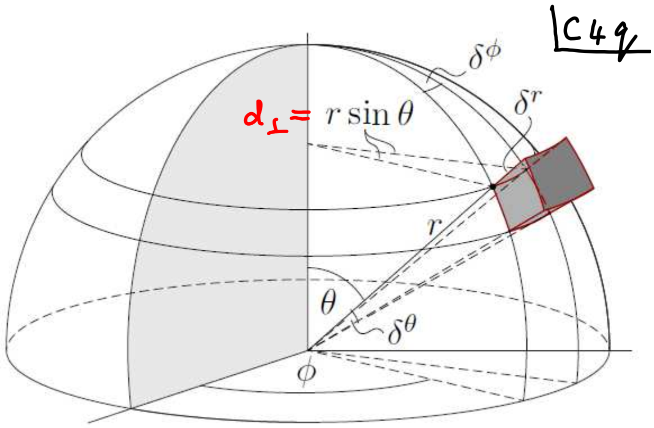
Volumen einer Kugel:

$$V = \int_0^R dr \, r^2 \int_0^\pi d\theta \, \sin\theta \int_0^{2\pi} d\phi = \frac{1}{3} R^3 \underbrace{\left[-\cos\theta\right]_0^\pi}_{2} 2\pi = \frac{4}{3} \pi R^3 \qquad (5)$$



Trägheitsmoment einer homogenen Kugel:

C4q



Dichte = ρ_0 =Masse/Volumen = $M / (\frac{4}{3}\pi R^3)$ (6)

Abstand v. Symmetrieachse: $d_\perp(\vec r) = r \sin\theta$

$$I = \int_V dV \, \overbrace{\rho(\vec r)}^{\rho_0} \, \overbrace{d_\perp^2(\vec r)}^{(r \sin\theta)^2}$$

$$= \int_0^R dr \, r^2 \int_0^\pi d\theta \, \sin\theta \int_0^{2\pi} d\phi \cdot \rho_0 \, r^2 (\sin\theta)^2$$

$$= \underbrace{\rho_0}_{\overset{(6)}{=} \frac{M}{4\pi/3 \, R^3}} \underbrace{\int_0^R dr \, r^4}_{\frac{1}{5} R^5} \int_0^\pi d\theta \, \underbrace{\sin\theta \left[1 - \sin^2\theta\right]}_{} \underbrace{\int_0^{2\pi} d\phi \, 1}_{2\pi}$$

$\cos\theta = u$: $$\int_{-1}^{1} du \, (1 - u^2) = \left[u - \frac{1}{3} u^3\right]_{-1}^{1} = 2\left[1 - \frac{1}{3}\right] = \frac{4}{3}$$

$$= \frac{2}{5} M R^2$$

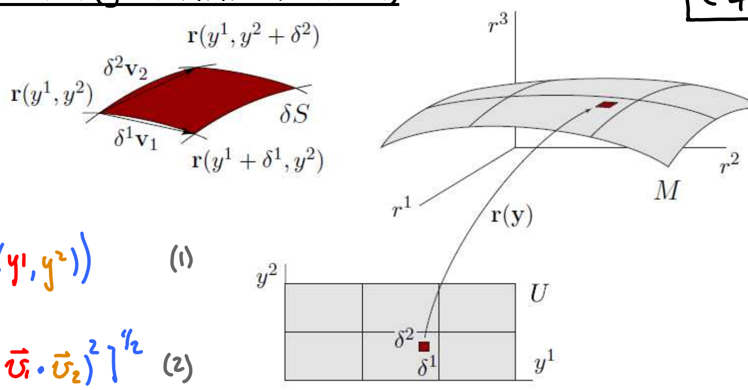
Standardsubstitution für Kugelkoordinaten:

$$\cos\theta = u$$
$$-\sin\theta = \frac{du}{d\theta}$$
$$\cos(0) = 1$$
$$\cos(\pi) = -1$$
$$\int_0^\pi d\theta \sin\theta = + \int_{-1}^{+1} du$$

C4.4 2D-Flächenintegrale in 3 Dimensionen (gekrümmte Flächen)

$$\vec{r}: \; U \subset \mathbb{R}^2 \to M \subset \mathbb{R}^3$$

$$\vec{y} = \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} \mapsto \vec{r}(\vec{y}) = \begin{pmatrix} x^1(\vec{y}) \\ x^2(\vec{y}) \\ x^3(\vec{y}) \end{pmatrix}$$



$$\int_M dS\, f(\vec{r}) \overset{(k.u)}{=} \int_U dy^1 dy^2 \, \| \partial_{y^1}\vec{r} \times \partial_{y^2}\vec{r} \| \, f(\vec{r}(y^1,y^2)) \qquad (1)$$

(analog zu Seite C4k)

$$\| \vec{v}_1 \times \vec{v}_2 \| = \left[(\vec{v}_1)^2 (\vec{v}_2)^2 - (\vec{v}_1 \cdot \vec{v}_2)^2 \right]^{1/2} \qquad (2)$$

Lagrange-Identität (L4k.7)

Für krummlinig-orthogonale Koordinaten, mit

$$\| \vec{v}_1 \times \vec{v}_2 \| = \| \vec{v}_1 \| \, \| \vec{v}_2 \| - 0 = \sqrt{g_{11}\, g_{22}} \qquad (3)$$

ist das Integrationsmaß:

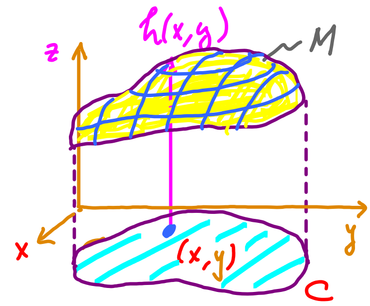
$$dS = dy^1 dy^2 \, v_1 v_2 = dy^1 dy^2 \sqrt{g_{11}\, g_{22}} \qquad (4)$$

Beispiel 1: Oberfläche einer Kugel (Radius R)

$$\boxed{dS \overset{(V2n.3)}{=} d\theta\, d\phi \cdot r \cdot r \sin\theta} \qquad (5)$$

$$A = \int_{\text{Kugel}} d\theta\, d\phi \cdot \left[r^2 \sin\theta \right]_{r=R} = R^2 \int_0^{\pi} d\theta \, \sin\theta \int_0^{2\pi} d\phi \cdot 1 = R^2 \cdot 2 \cdot 2\pi = 4\pi R^2 \qquad (6)$$

Beispiel 2: Hügel $\qquad y^1 = x, \quad y^2 = y$



$$\vec{r}(x,y) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ h(x,y) \end{pmatrix}, \qquad (1)$$

'Höhe des Hügels'

$$\vec{v}_1 = \partial_x \vec{r} = \begin{pmatrix} 1 \\ 0 \\ h_x \end{pmatrix}, \; h_x \equiv \frac{\partial h}{\partial x} \qquad \vec{v}_2 = \partial_y \vec{r} = \begin{pmatrix} 0 \\ 1 \\ h_y \end{pmatrix}, \; h_y \equiv \frac{\partial h}{\partial y} \qquad (2)$$

$$\| \partial_x \vec{r} \times \partial_y \vec{r} \| = \| \vec{v}_1 \times \vec{v}_2 \| \overset{(r.2)}{=} \left[(\vec{v}_1)^2 (\vec{v}_2)^2 - (\vec{v}_1 \cdot \vec{v}_2)^2 \right]^{1/2} \qquad (3)$$

$$\overset{(1)}{=} \left[(1 + h_x^2)(1 + h_y^2) - (\underbrace{h_x h_y}_{\neq 0})^2 \right]^{1/2} = \left[1 + h_x^2 + h_y^2 \right]^{1/2} \qquad (4)$$

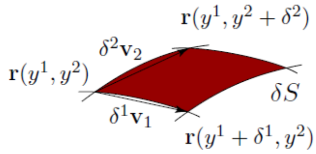
(also nicht-orthog. Koordinatensystem)

Wie groß ist 'Hügeloberfläche' über dem Bereich $(x,y) \in C$?

$$A_M = \int_M dS \overset{(r.1)}{=} \int_C dx\, dy \, \| \partial_x \vec{r} \times \partial_y \vec{r} \| \overset{(4)}{=} \int_C dx\,dy \left[1 + h_x^2 + h_y^2 \right]^{1/2} \qquad (5)$$

Zusammenfassung: 2D-Flächenintegrale für Fläche in d = 2,3 Dimensionen

$$\vec r: \mathcal{U}\subset\mathbb{R}^2 \to M\subset\mathbb{R}^n, \qquad \vec y = \begin{pmatrix} y^1\\ y^2\end{pmatrix} \mapsto \vec r(\vec y) = \begin{pmatrix} x^1(\vec y)\\ \vdots \\ x^n(\vec y)\end{pmatrix} \tag{1}$$

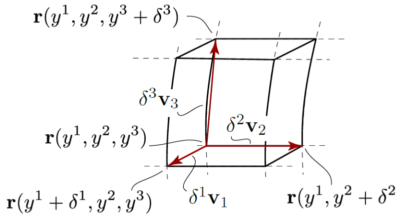


$$\int_M dS\, f(\vec r) = \int_{\mathcal U} dy^1 \int dy^2 \,\|\partial_{y^1}\vec r\times\partial_{y^2}\vec r\|\, f(\vec r(y^1,y^2)) \tag{2}$$

für krummlinig-orthogonale Koord.: $\|\partial_{y^1}\vec r\times\partial_{y^2}\vec r\| = v_{y^1}v_{y^2} = \sqrt{g_{11}\,g_{22}}$ (3)

Integrationsmaß: Polar: $dS = \rho\, d\rho\, d\phi$ (4) Kugel: $dS = d\theta\, d\phi\, r^2\cdot\sin\theta$ (5)

Zusammenfassung: 3D Volumenintegrale



$$\vec r: \mathcal{U}\subset\mathbb{R}^3 \to V\subset\mathbb{R}^3, \qquad \vec y \mapsto \vec r(\vec y) = \begin{pmatrix} x^1(\vec y)\\ x^2(\vec y) \\ x^3(\vec y)\end{pmatrix} \tag{6}$$

$$\int_V dr^1\, dr^2\, dr^3\, f(\vec r) = \int_{\mathcal U} dy^1\, dy^2\, dy^3\, \|(\partial_{y^1}\vec r\times\partial_{y^2}\vec r)\cdot\partial_{y^3}\vec r\|\, f(\vec r(y^1,y^2,y^3)) \tag{7}$$

für krummlinig-orthogonale Koord.: $\|(\partial_{y^1}\vec r\times\partial_{y^2}\vec r)\cdot\partial_{y^3}\vec r\| = v_{y^1}\,v_{y^2}\,v_{y^3} = \sqrt{g_{11}\,g_{22}\,g_{33}}$ (8)

Integrationsmaß: Zylinder: $dV = \rho\, d\rho\, d\phi\, dz$ (9) Kugel: $dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$ (10)