

Inverse problems and machine learning in medical physics

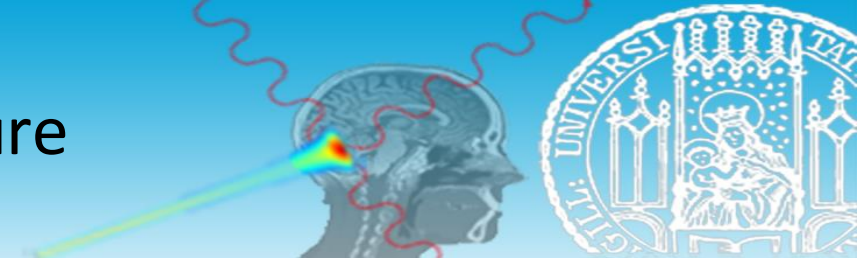
Numerical image reconstruction

Dr. Chiara Gianoli

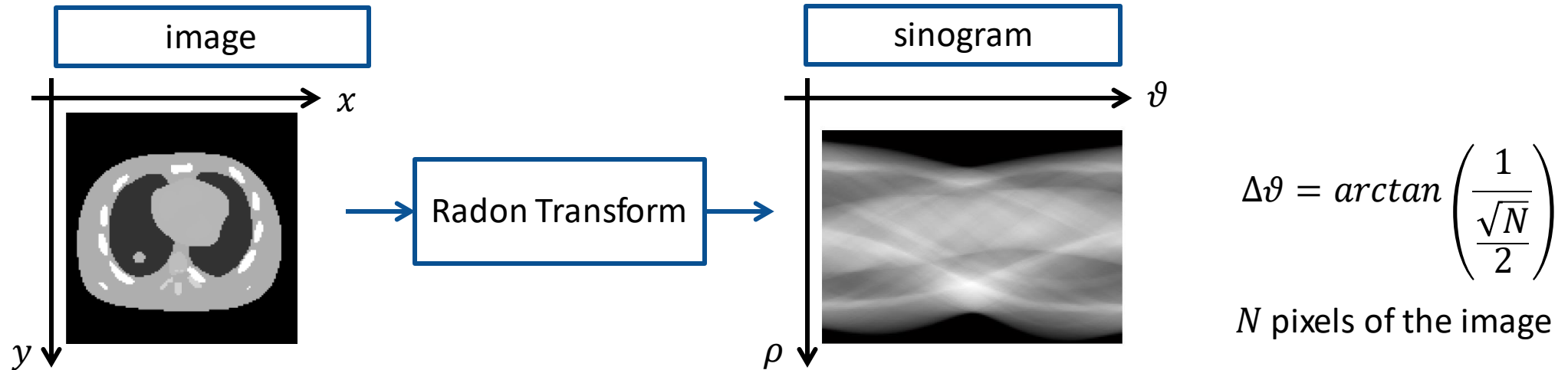
15/11/2022

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Outlook from the previous lecture



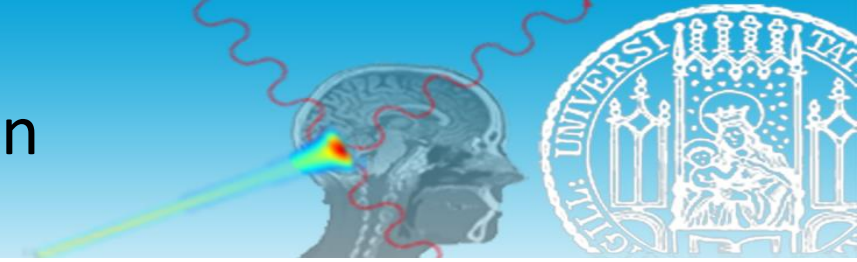
- Analytical image reconstruction is based on the **continuous form** of the **Radon Transform**



- The **Fourier Slice Theorem**, provided with the **Nyquist theorem of sampling**, enables the implementation and application of **analytical reconstruction algorithms** (i.e., the filtered back-projection)

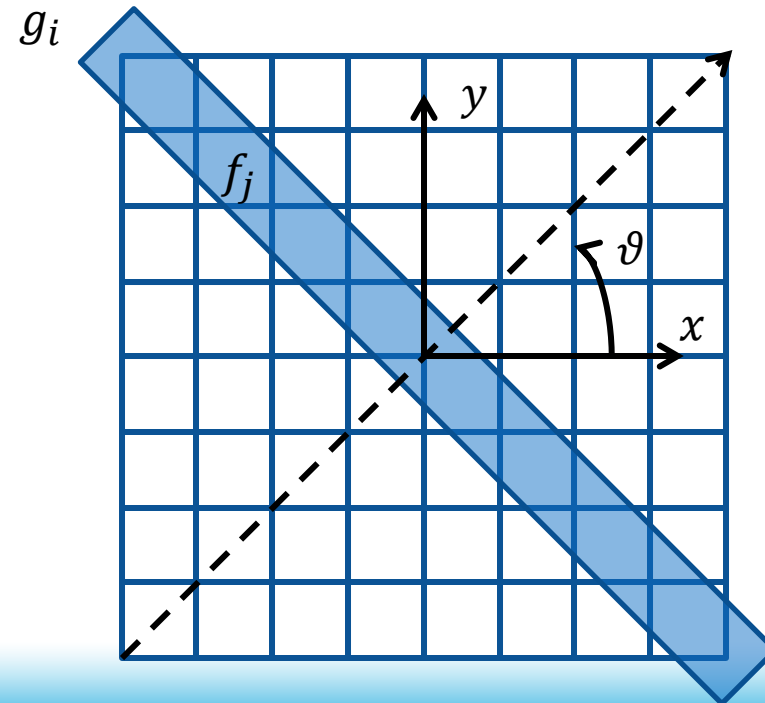
$$\hat{f}_\rho(w_x, w_y) = \int_{-\infty}^{+\infty} R(f) e^{-2\pi i(\rho w_\rho)} d\rho = \hat{R}(w_\rho)$$

Numerical image reconstruction

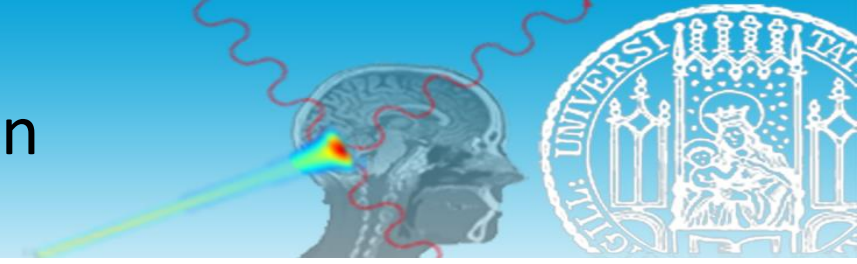


- Numerical image reconstruction does not rely on the **Fourier Slice Theorem** and the **Nyquist theorem of sampling**
 - The image and the sinogram have not necessarily to be continuous, thus enabling the reconstruction in presence of **geometrical constraints**
- Numerical image reconstruction can be described as the solution of a **linear system of equations**
 - I equations, one for each projection
 - J unknowns, one for each pixel

$$\begin{cases} a_{11}f_1 + a_{12}f_2 + \dots + a_{1J}f_J = g_1 \\ \dots \\ a_{I1}f_1 + a_{I2}f_2 + \dots + a_{IJ}f_J = g_I \end{cases}$$



Numerical image reconstruction



- Numerical image reconstruction requires the description of the **imaging system model** in terms of geometry of the integration lines

$$\bar{g}_i = \sum_{j=1}^J a_{ij} f_j$$

$$g_i = \bar{g}_i + \text{noise}$$

Model of the
projections

Measurement
(projections)

- The **imaging system model** is described in the **system matrix** of the numerical reconstruction $A = \{a_{ij}\}$, whose size is $I \times J$
 - The integration line is traced as intersection **length/area/volume** with the image **pixels/voxels**

The system matrix



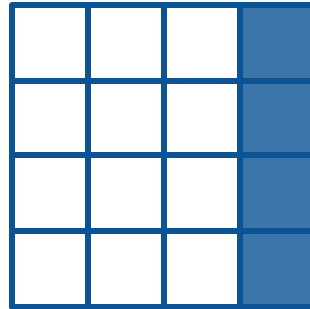
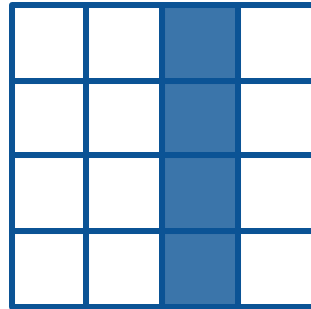
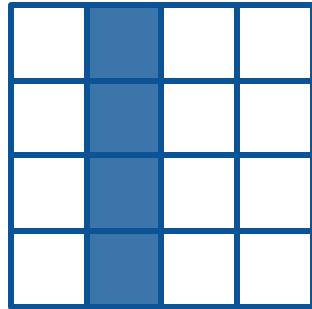
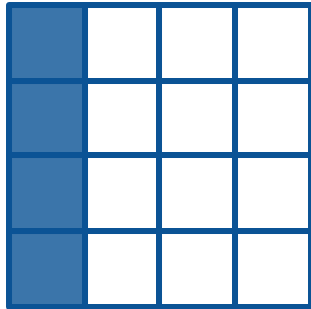
$\rho = 1$

$\rho = 2$

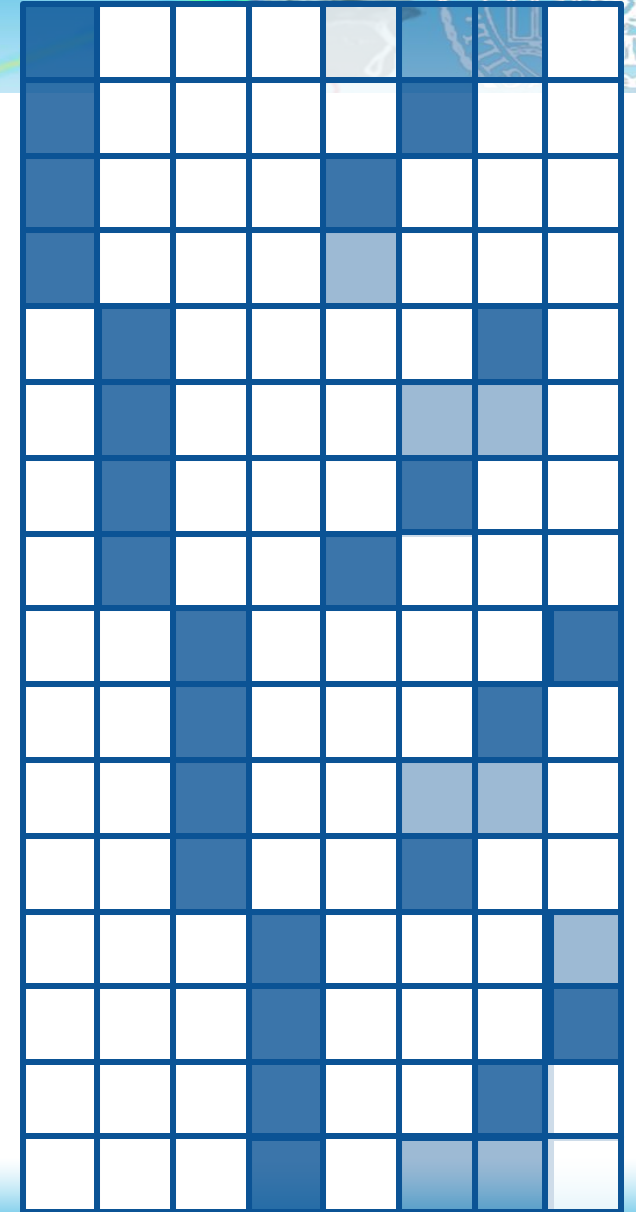
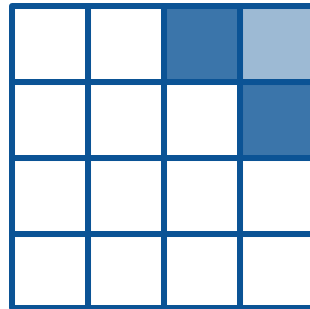
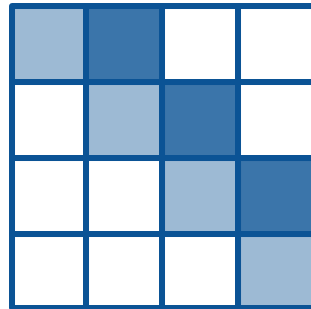
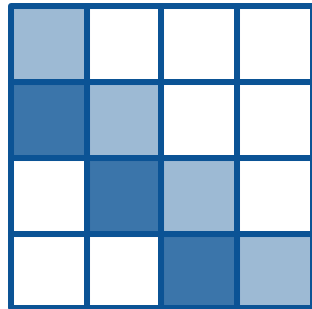
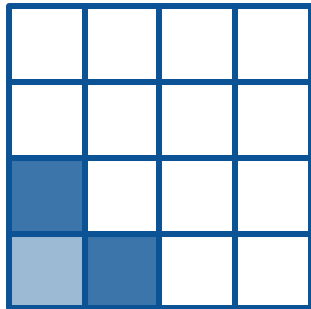
$\rho = 3$

$\rho = 4$

$\vartheta = 0$
degree

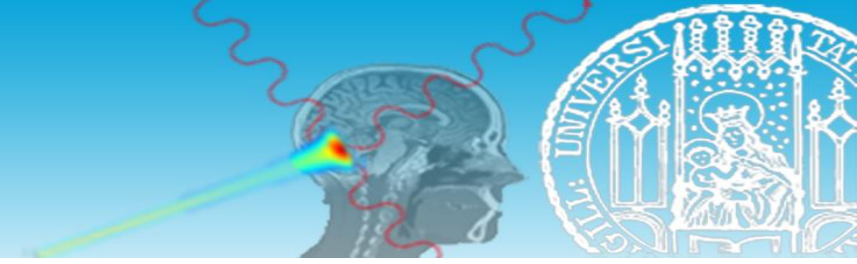


$\vartheta = 45$
degrees



- Matrix construction for each integration line and for each projection angle
- Vectorization or linearization of this matrix
- Assembly in the system matrix
- System matrix transposition

The system matrix



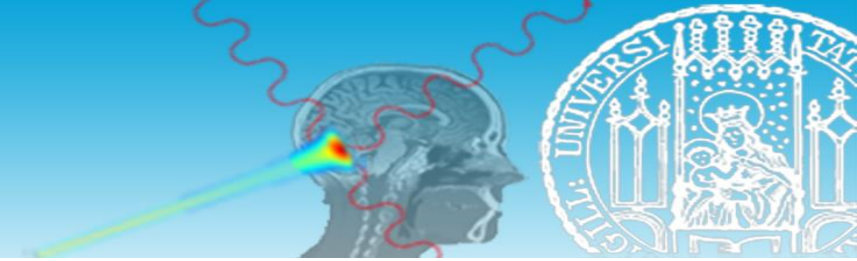
- The **forward-projection** of the image is calculated as a **matrix-vector product** of the system matrix and the “vectorized” image

One column for each projection

A^T	One row for each pixel	a_{11}	a_{21}			...				a_{I1}	
		a_{1J}	a_{2J}								a_{IJ}

$$g = Af \quad \begin{cases} a_{11}f_1 + a_{12}f_2 + \dots + a_{1J}f_J = g_1 \\ \dots \\ a_{I1}f_1 + a_{I2}f_2 + \dots + a_{IJ}f_J = g_I \end{cases}$$

The system matrix



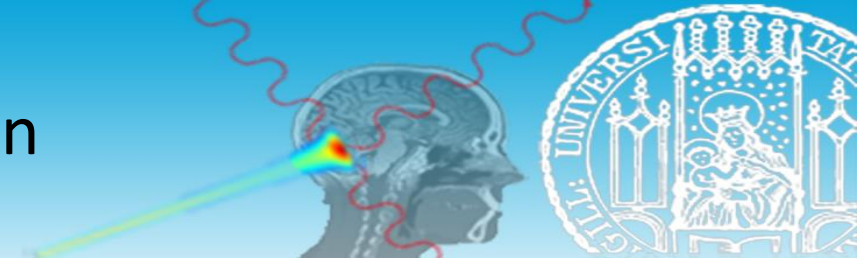
- The **back-projection** of the sinogram is calculated as a **matrix-vector product** of the transposed system matrix and the “vectorized” sinogram

One column for each projection

A^T One row for each pixel	a_{11}	a_{21}			...			a_{I1}
	a_{1J}	a_{2J}						

$$f = A^T g \quad \begin{cases} a_{11}g_1 + a_{21}g_2 + \dots + a_{I1}g_I = f_1 \\ \dots \\ a_{1J}g_1 + a_{2J}g_2 + \dots + a_{IJ}g_I = f_J \end{cases}$$

Numerical image reconstruction

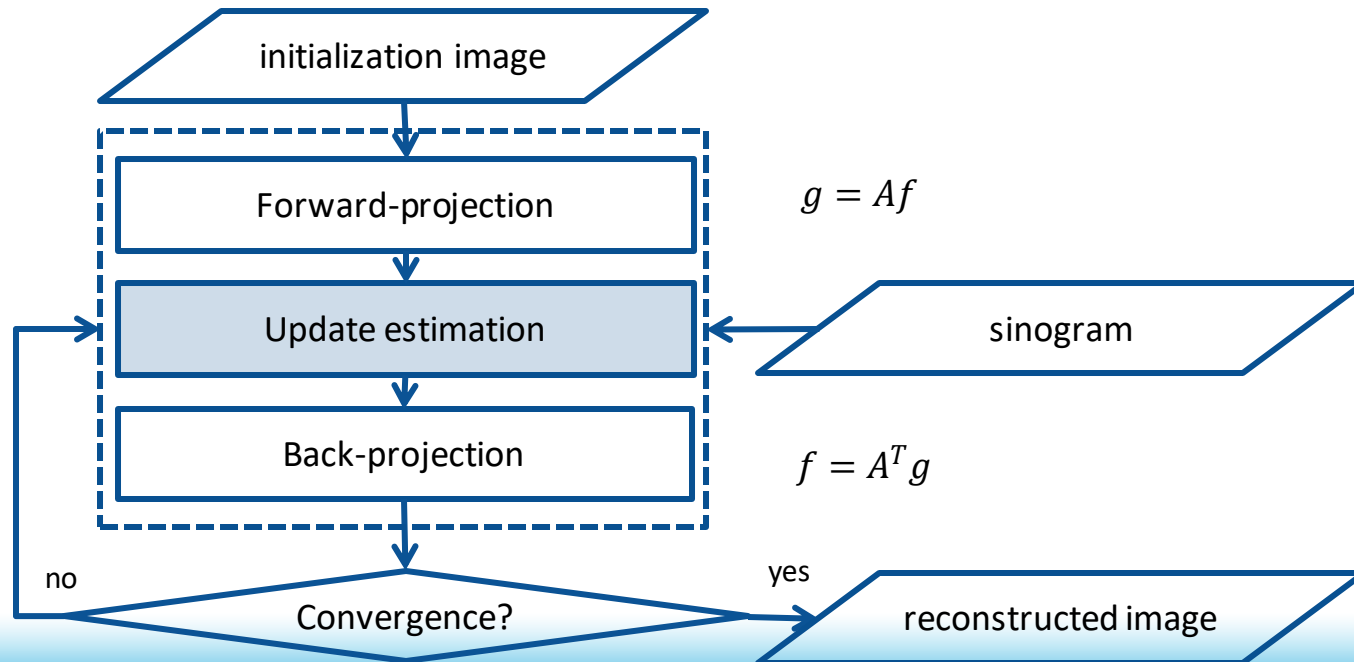


- Several computational methods aim at solving linear systems of equations
 - Least square optimization for overdetermined system of equations (more equations than unknowns)

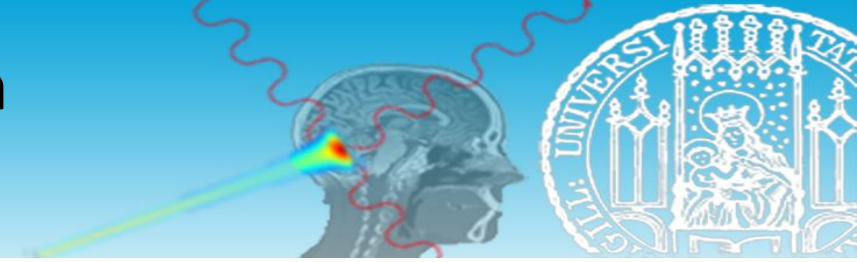
$$f_{min} = \operatorname{argmin} \left\| g_i - \sum_{j=1}^J a_{ij} f_j \right\|^2 \qquad f_{min} = (A^T A)^{-1} A^T g_i$$

- Numerical (iterative) optimization

$$f_{min} = \operatorname{argmin}_{f_j} F(\bar{g}_i, g_i)$$



Principle of X-ray transmission tomography



- The physical properties of the object of interest cause the **attenuation of X-ray beams**, described through attenuation coefficients μ , assumed to be constant for different energies (or assuming a mono-energetic X-ray beam)
 - The projection expresses the **intensity reduction** due to photon attenuation in the object of interest
 - The attenuation is described by **Lambert Beer's law**:

$$I(x + \Delta x) = I(x) - \mu(x)I(x)\Delta x \qquad \frac{dI}{dx} = -\mu(x)I(x) \qquad \boxed{\text{differential equations}}$$

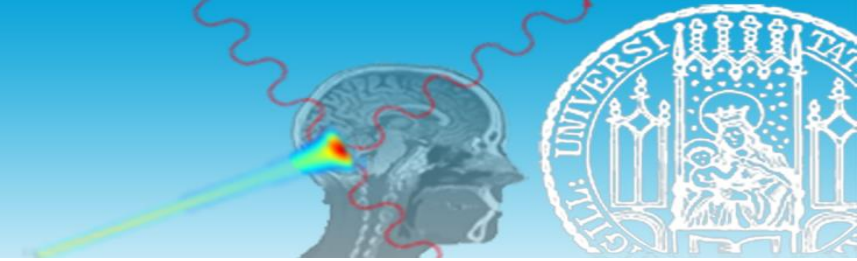
- The tomographic image reconstruction of the attenuation coefficients μ is enabled by the Lambert Beer's law that models the **projection** as a **line integral** of the physical variables

$$-\ln \frac{I(x = X)}{I(x = 0)} = \int_0^X \mu(x) dx$$

integral equation

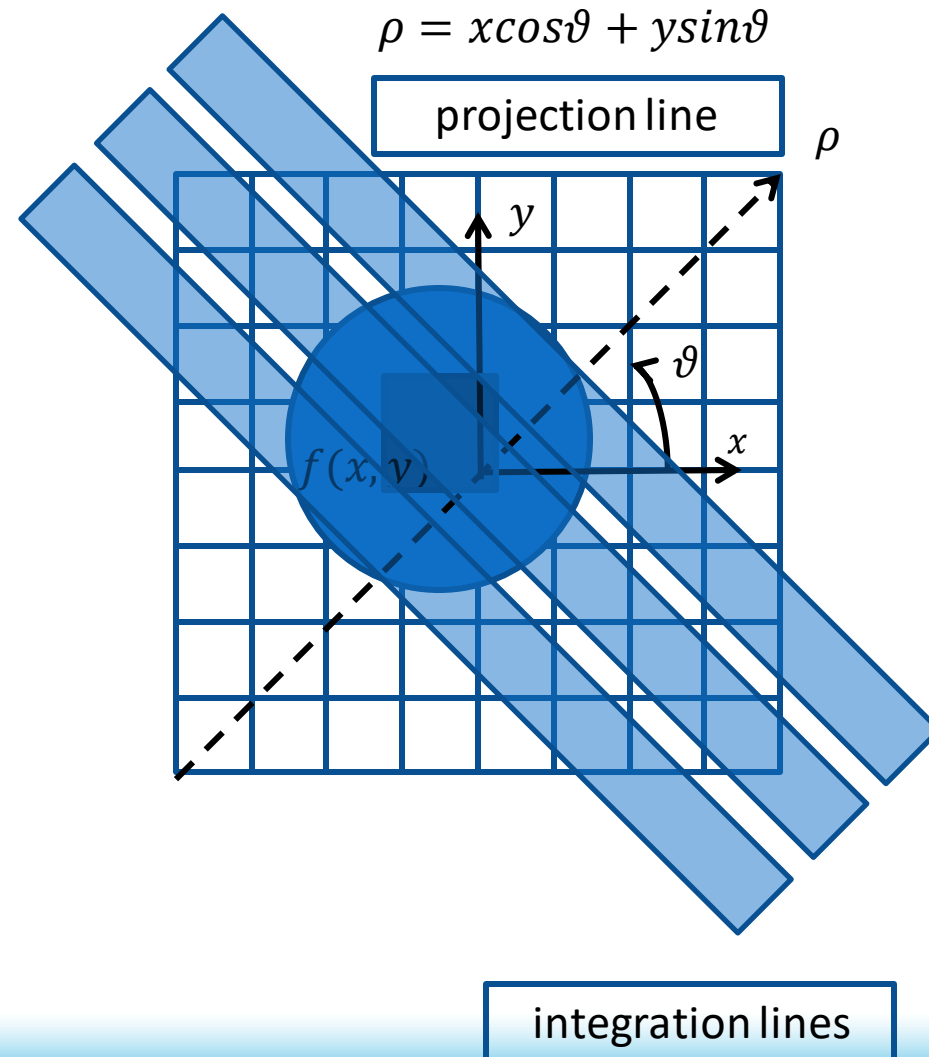
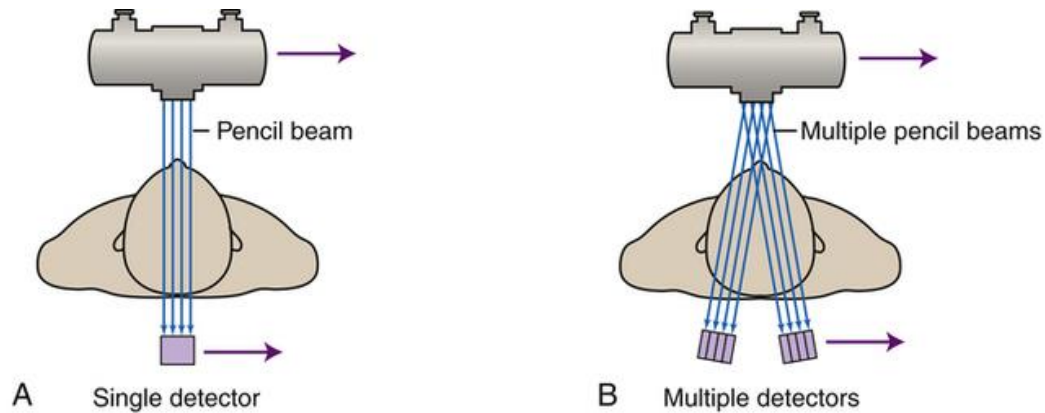
$$\int \frac{1}{x} dx = \ln(x)$$

Imaging system geometry in X-ray CT

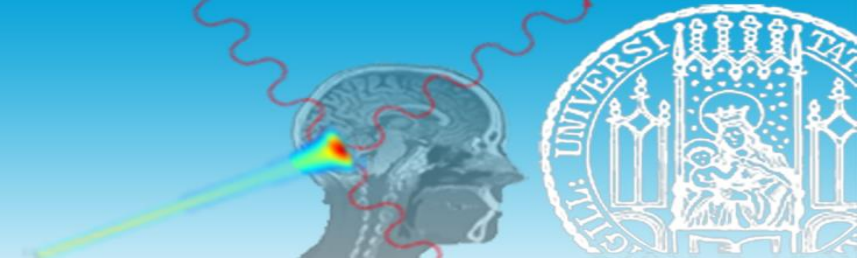


Parallel geometry

- Infinite center of projection
- Fixed projection angle for the projection line, parallel integration lines

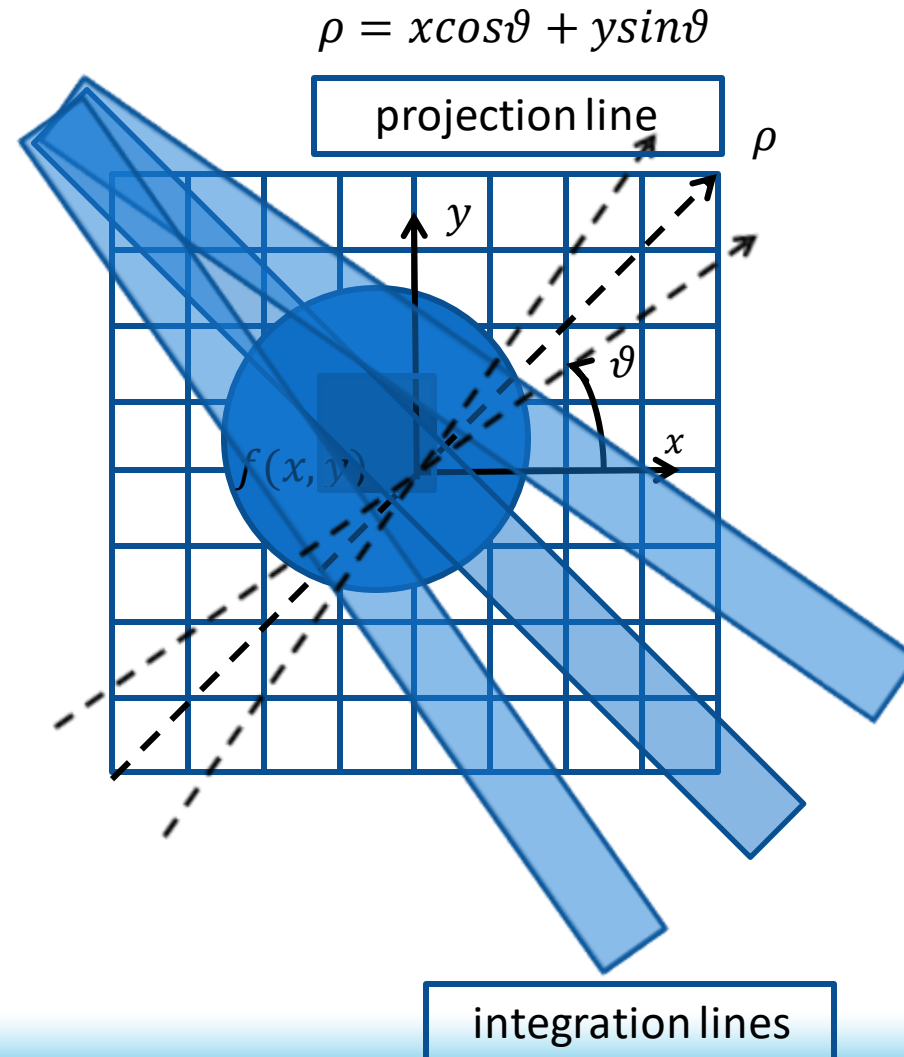
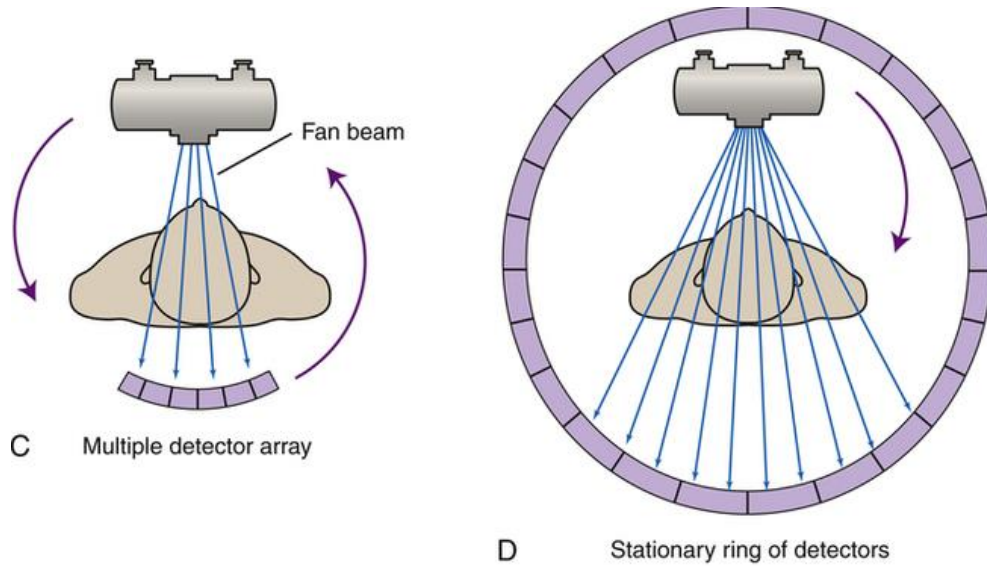


Imaging system geometry in X-ray CT

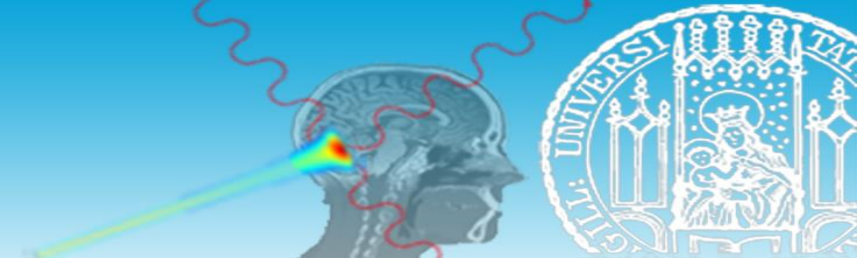


Fan geometry

- Finite center of projection
- Variable projection angle for the projection line, diverging integration lines

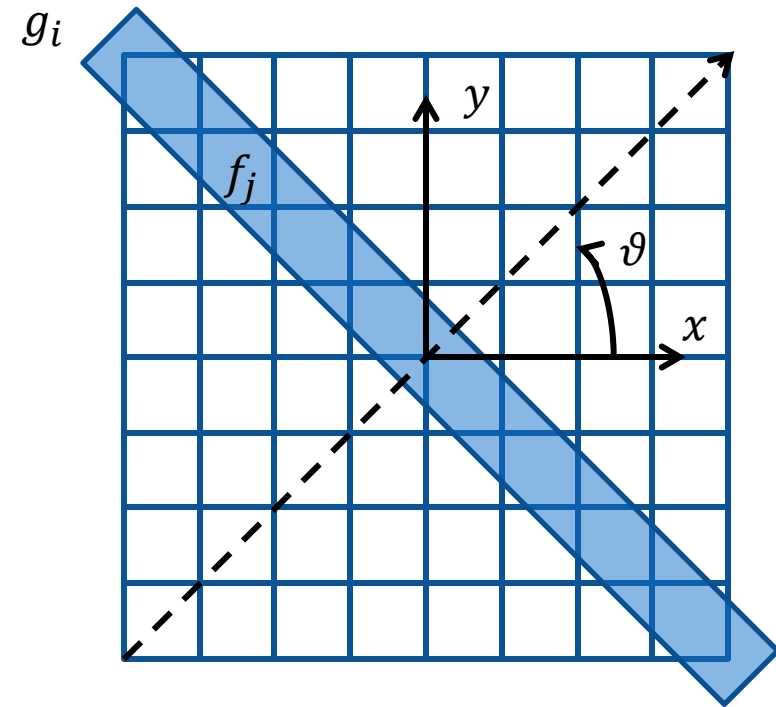


Algebraic Reconstruction Technique

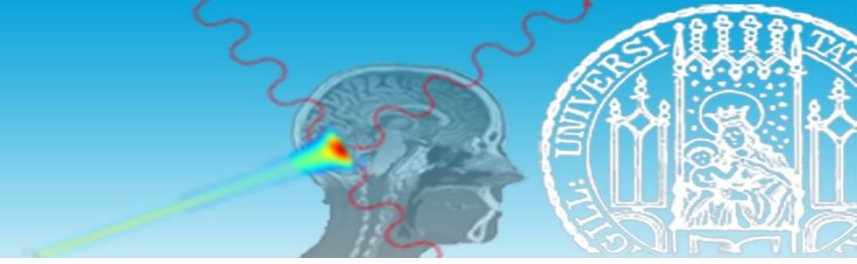


- The reconstruction consists in solving a system of I equations, where I is the number of boundaries (the number of **projections**), relying on the constants a_{ij} describing the imaging system model (the **system matrix**)
- Each projection is interpreted as an **hyperplane** in a **J -dimensional space**, where J is the degrees of freedom (the number of pixels/voxels of the image and thus, the number of the unknowns)
 - If existing, the intersection of the I hyperplanes represents the solution of the system of equations
 - The dimension of the hyperplane is **$J-1$**

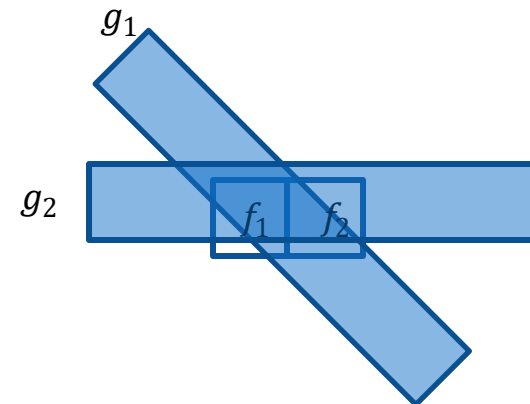
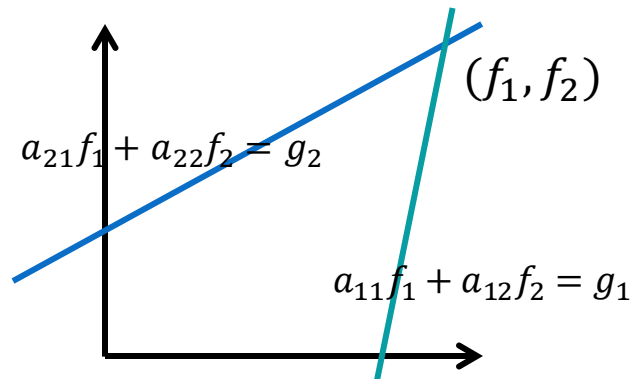
$$\begin{cases} a_{11}f_1 + a_{12}f_2 + \dots + a_{1J}f_J = g_1 \\ \dots \\ a_{I1}f_1 + a_{I2}f_2 + \dots + a_{IJ}f_J = g_I \end{cases}$$



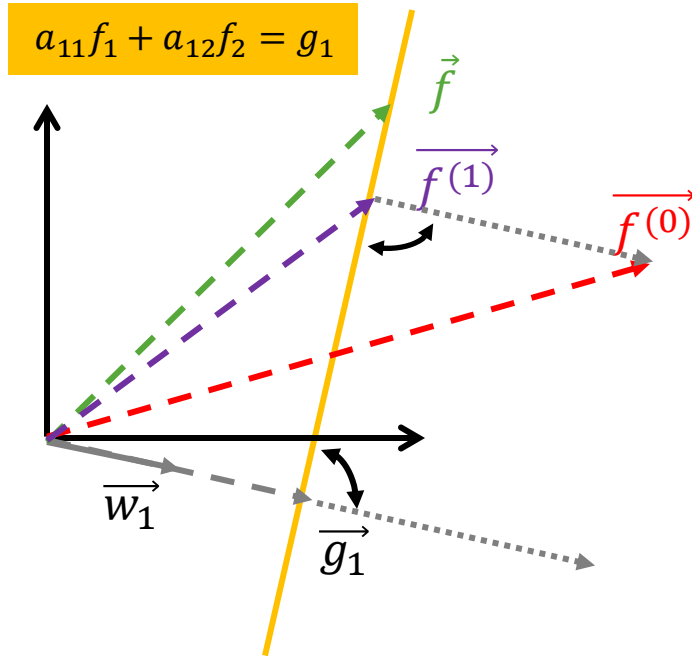
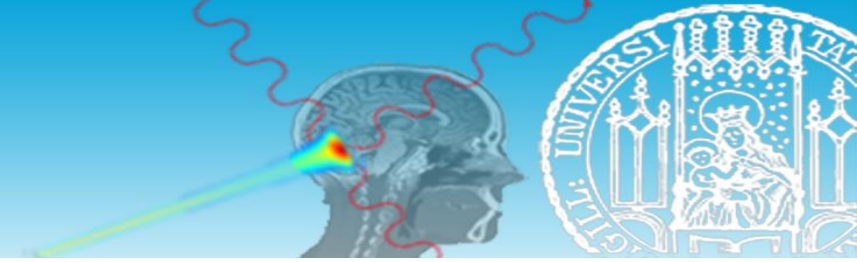
Algebraic Reconstruction Technique



- Simplified imaging system geometry: 2 unknowns and 2 parameters
 - The lines, or 1-dimensional hyperplanes, represent the boundaries (i.e. the projections)
 - The intersection point represent the solution (i.e. the image)



Algebraic Reconstruction Technique



- Describe the projection g_1 as a line (i.e., the yellow line)
- Define the vector \vec{a}_1 as the 2 coefficients of the system matrix relevant to the projection

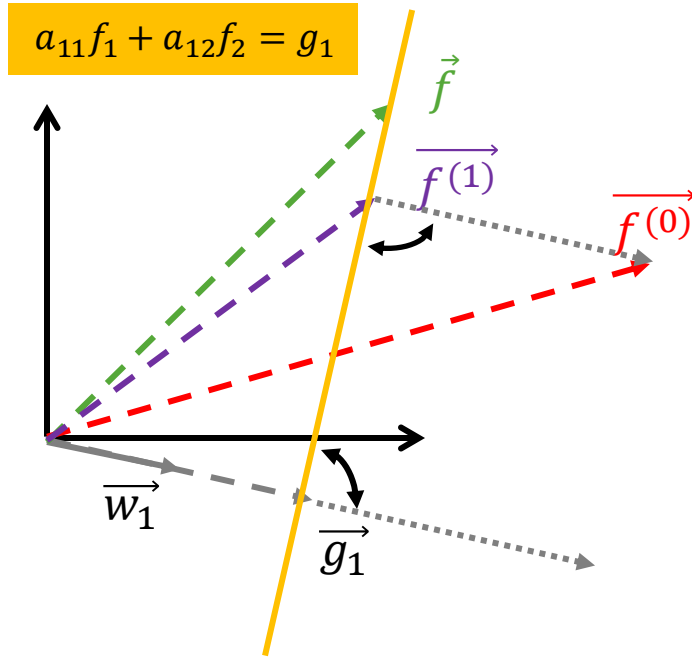
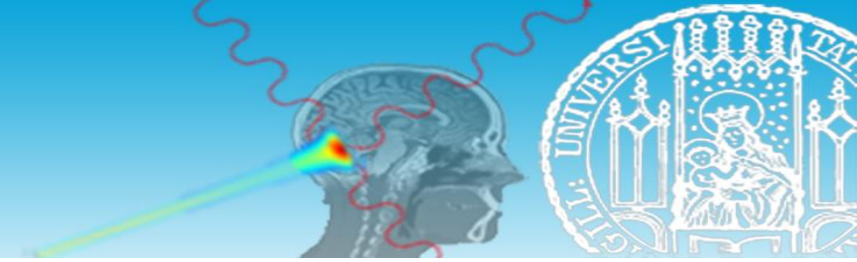
$$\vec{a}_1 = (a_{11}, a_{12})$$
- Express the unit vector \vec{w}_1 as the vector \vec{a}_1 (perpendicular to the line by construction) divided by its modulus

$$\vec{w}_1 = \frac{\vec{a}_1}{\sqrt{\vec{a}_1 \cdot \vec{a}_1}}$$

- Define $g_1 = |\vec{g}_1|$ as the scalar product (dot product) of an arbitrary vector \vec{f} laying on the line by the unit vector \vec{w}_1 (or project an arbitrary vector \vec{f} laying on the line along \vec{w}_1)

$$g_1 = |\vec{g}_1| = \vec{f} \cdot \vec{w}_1 = \frac{\vec{a}_1 \cdot \vec{f}}{\sqrt{\vec{a}_1 \cdot \vec{a}_1}}$$

Algebraic Reconstruction Technique



- Define $\vec{f}^{(0)}$ as a vector not laying on the line
- Define $\vec{f}^{(1)}$ as a vector laying on the line, as the solution lays on the line
- Express the update vector as difference between $\vec{f}^{(1)}$ and $\vec{f}^{(0)}$

$$\vec{f}^{(0)} - \vec{f}^{(1)} = (\vec{f}^{(0)} \cdot \vec{w}_1 - g_1) \cdot \vec{w}_1$$

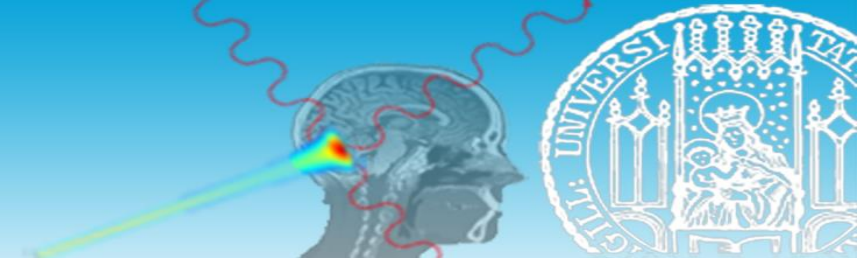
$$\vec{f}^{(1)} = \vec{f}^{(0)} - (\vec{f}^{(0)} \cdot \vec{w}_1 - g_1) \cdot \vec{w}_1$$

- Substitute \vec{w}_1

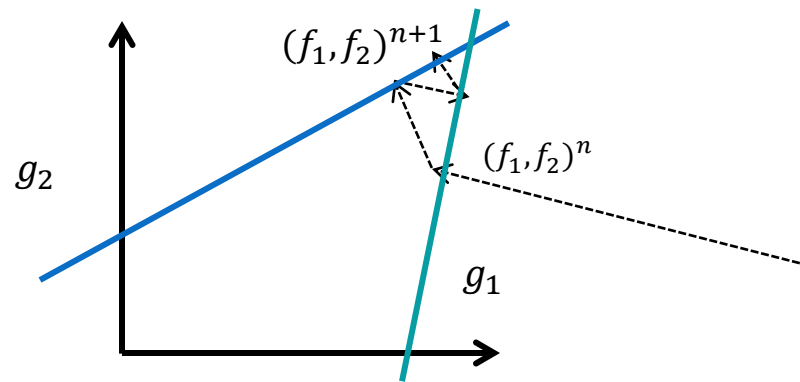
$$\vec{f}^{(1)} = \vec{f}^{(0)} - (\vec{f}^{(0)} \cdot \frac{\vec{a}_1}{\sqrt{\vec{a}_1 \cdot \vec{a}_1}} - g_1) \cdot \frac{\vec{a}_1}{\sqrt{\vec{a}_1 \cdot \vec{a}_1}}$$

$$\vec{f}^{(1)} = \vec{f}^{(0)} - \frac{(\vec{f}^{(0)} \cdot \vec{a}_1 - g_1)}{\vec{a}_1 \cdot \vec{a}_1} \cdot \vec{a}_1$$

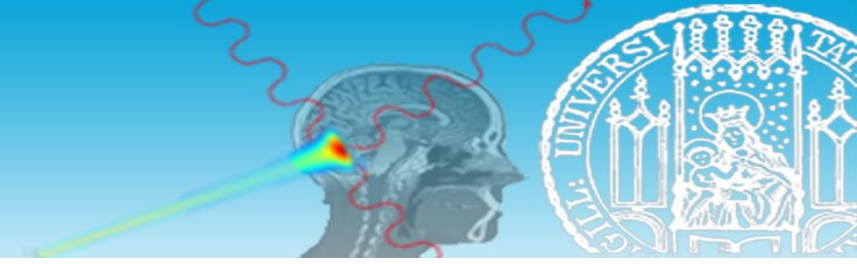
Algebraic Reconstruction Technique



- The update vector moves perpendicularly within boundaries (the [Kaczmarz method](#))
- Additive update of the image, after the projection has been considered (projection line per projection line)
- One iteration of ART is completed when all the projections have been considered

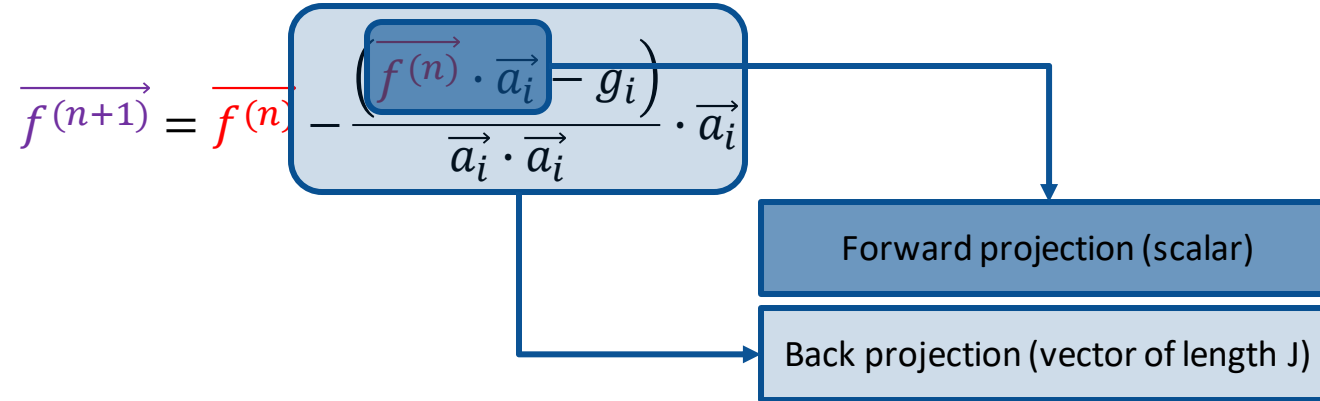


Algebraic Reconstruction Technique (ART)



- The update vector is calculated for each projection i , defined by \vec{a}_i

$$a_i = (a_{i1}, a_{i2}, \dots, a_{iN})$$



Initialization $\vec{f}^{(0)}$

for it = 1: nIterations

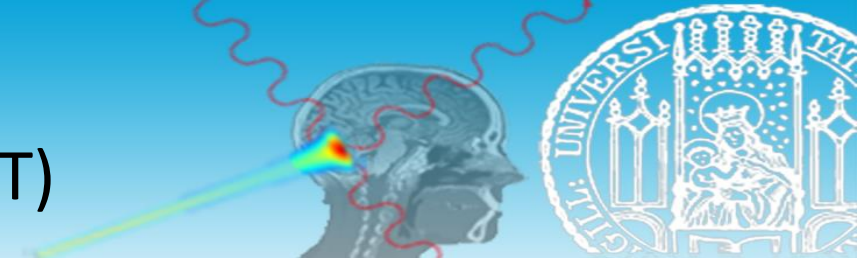
 for i = 1: I

 update estimation

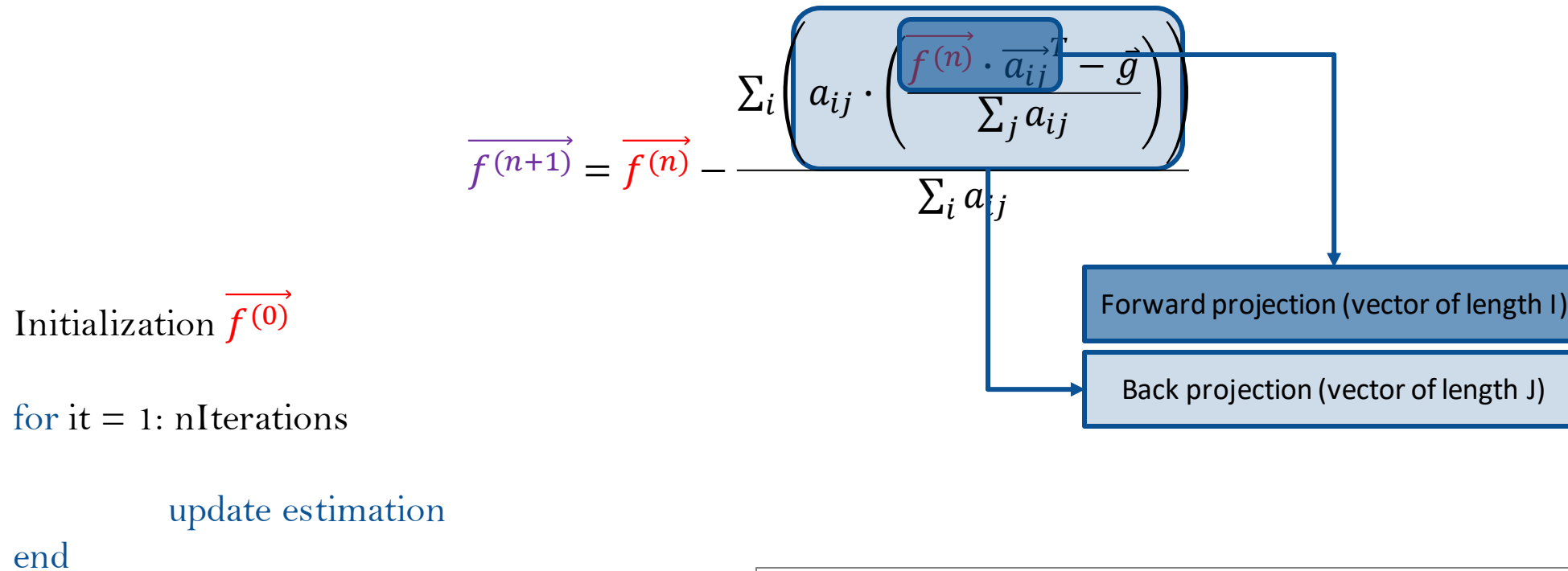
 end

end

Simultaneous Algebraic Reconstruction Technique (SART)

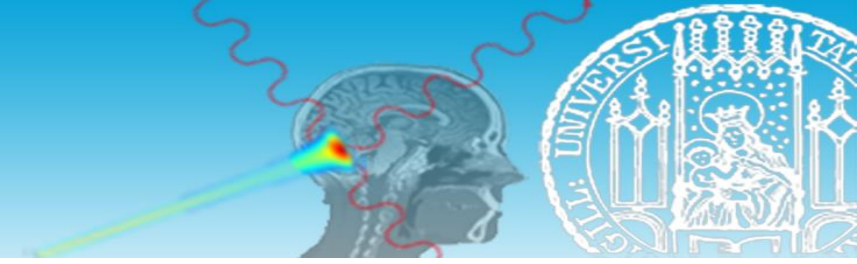


- Additive update of the image, contemporaneous for all the projection lines
- Under ideal conditions, one iteration of SART coincides with I updates of the ART

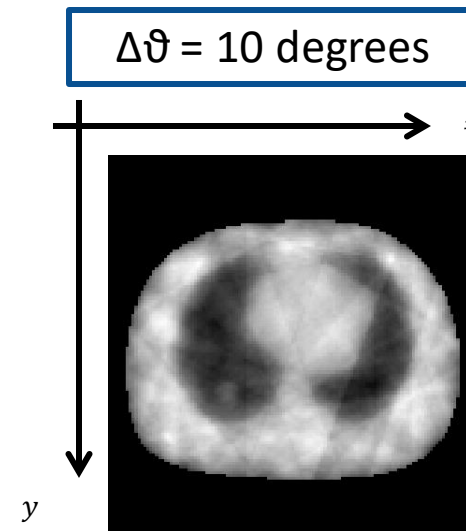
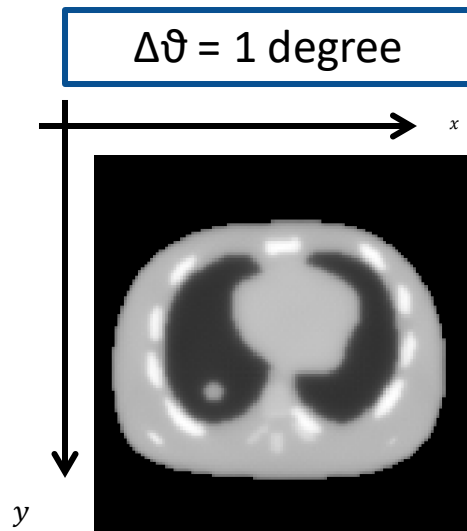


Andersen, A. H., & Kak, A. C. (1984). Simultaneous algebraic reconstruction technique (SART): a superior implementation of the ART algorithm. *Ultrasonic imaging*, 6(1), 81-94.

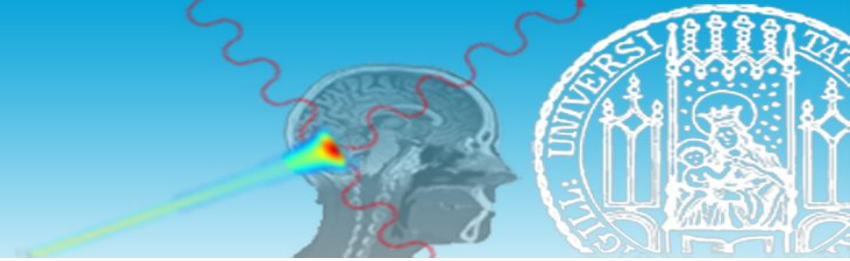
Simultaneous Algebraic Reconstruction Technique



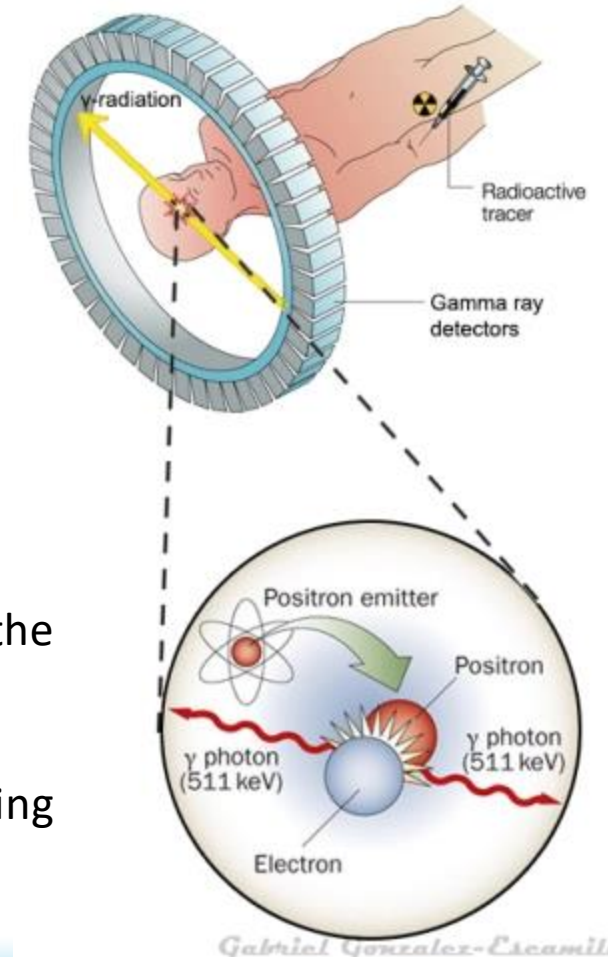
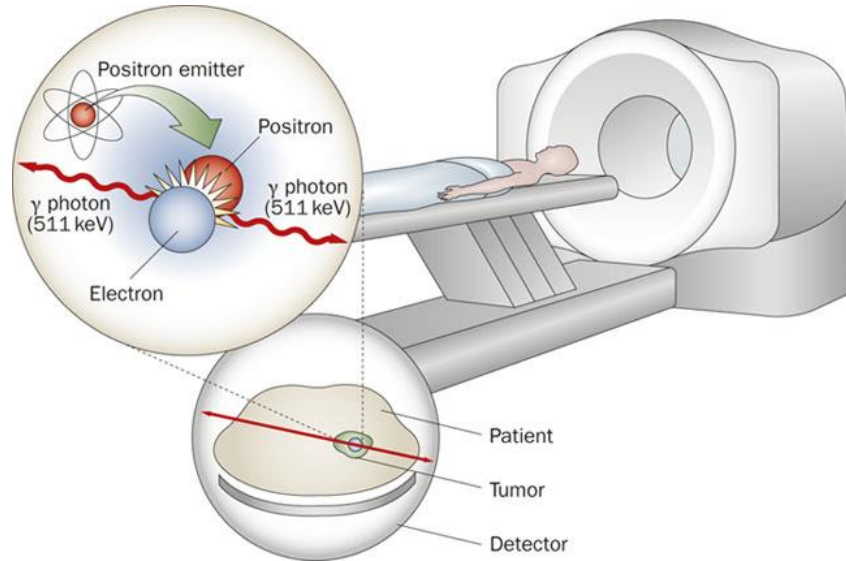
- Image reconstructed according to Simultaneous Algebraic Reconstruction Technique (SART) by setting the number of projection angles $n\vartheta = 180$ with spacing $\Delta\vartheta = 1$ degree and 10 degrees, the number of projection lines $n_p = 128$ and the number of iterations equal to 40



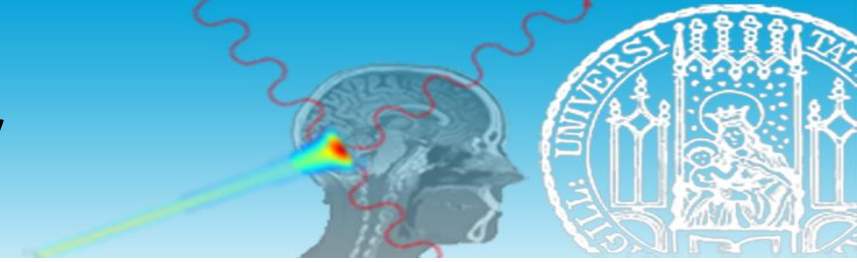
Principle of positron emission tomography



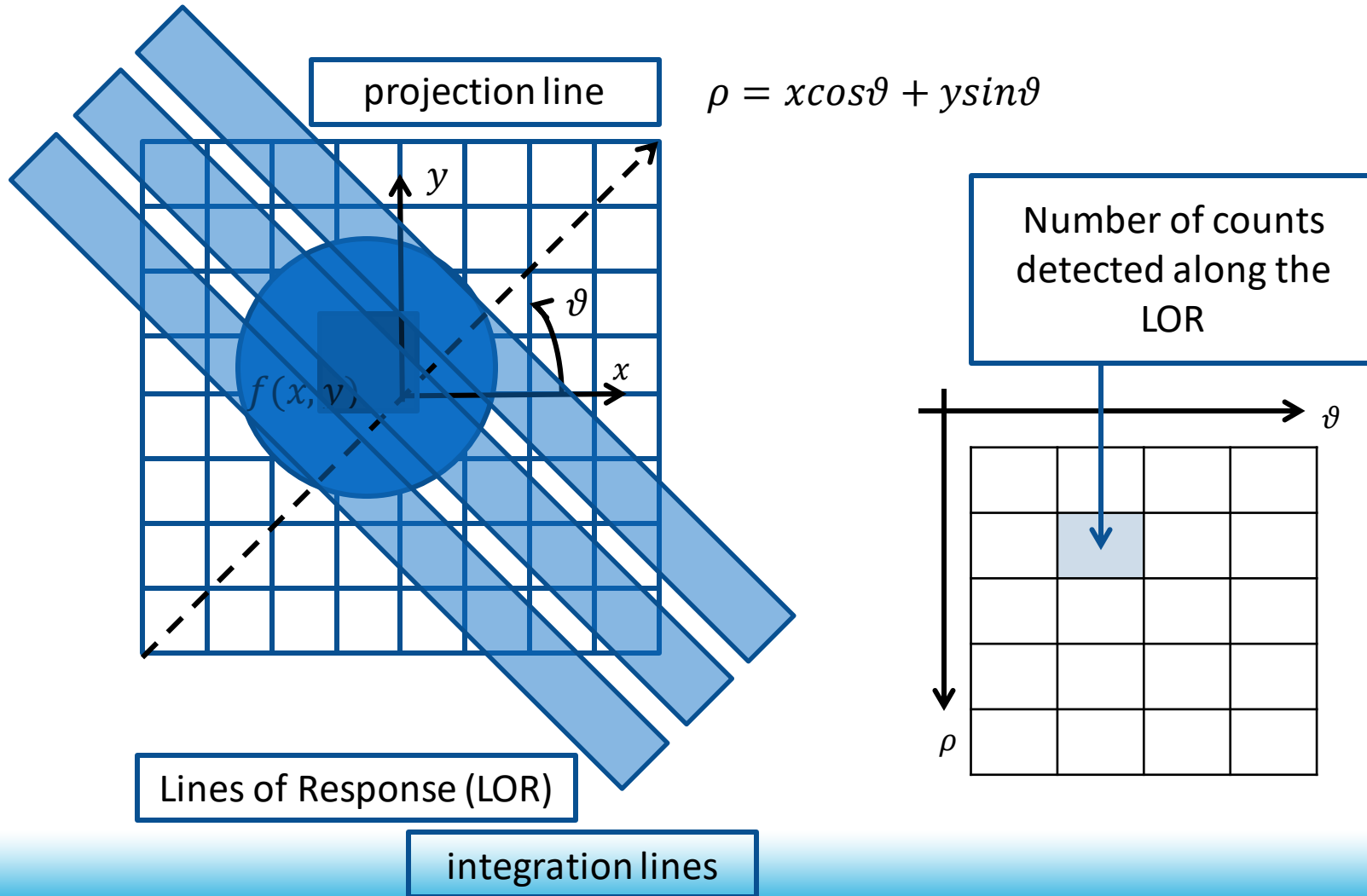
- The **energy source** is the β^+ **emitter** and concentrates in the object of interest due to biological properties of the **radiotracer**
- The physical effect relevant to PET imaging is the β^+ **emission** inside the object of interest
- Two synchronized crystals detect the **two annihilation photons** in **time coincidence** ("the count")
- The **integration line** is defined along the **Line of Response** (LOR) as the line along which the count is detected
- The **projection line** is defined by parallel (co-planar or not) **integration lines** connecting different crystals



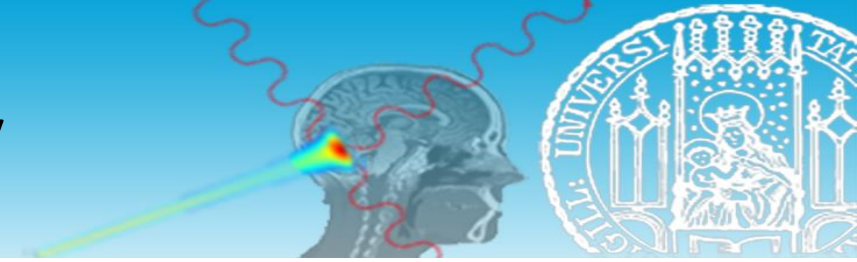
The imaging system geometry



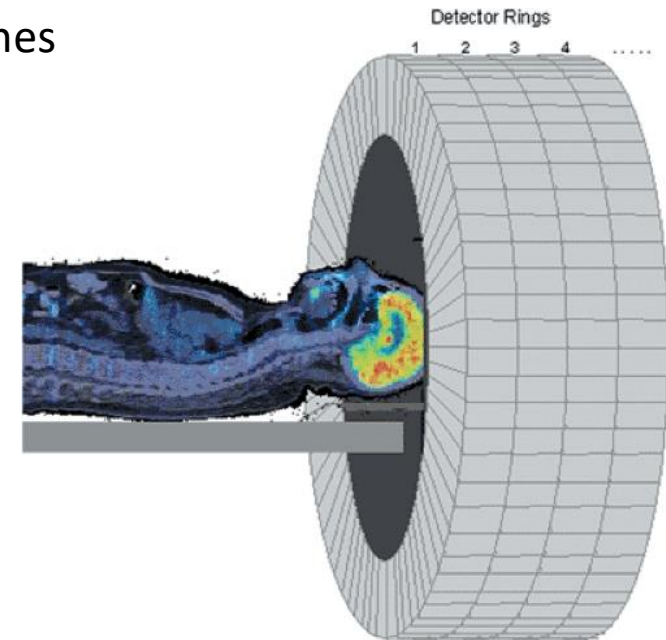
- The counts are organized in the **sinogram**, as a function of projection lines and projection angles



The imaging system geometry



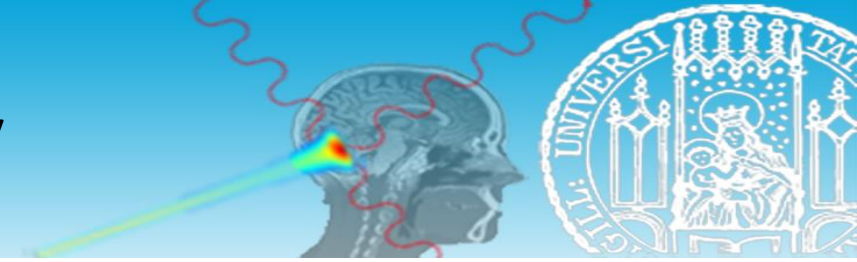
- The PET scanner is composed by several rings of crystals that can be synchronized or not
 - The synchronization of crystals belonging to the same ring defines the **direct sinogram**
 - The projection line is defined by parallel and co-planar integration lines
 - The sinogram is defined by co-planar projection lines
 - The synchronization of crystals belonging to different rings defines an **oblique sinogram**
 - The projection line is defined by parallel but not co-planar integration lines




<https://radiologykey.com>

1 2 3 4

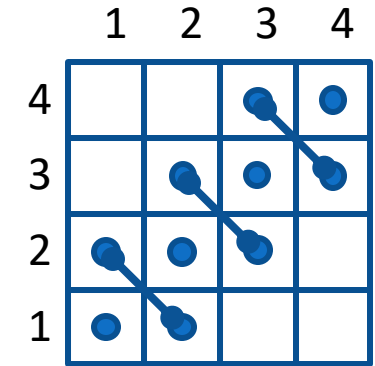
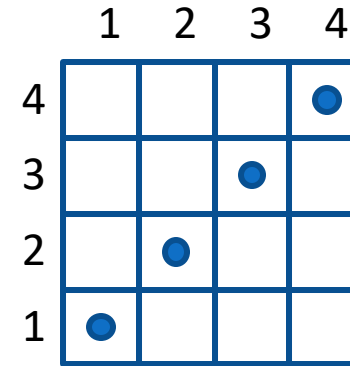
The imaging system geometry



- The “michelogram” displays the crystal synchronization and thus, the acquisition/reconstruction modality

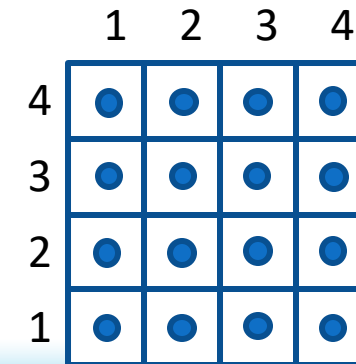
- The “2D-mode” is defined for **direct sinograms** and **averaged oblique sinograms** ( = average)

- 2D tomographic image reconstruction

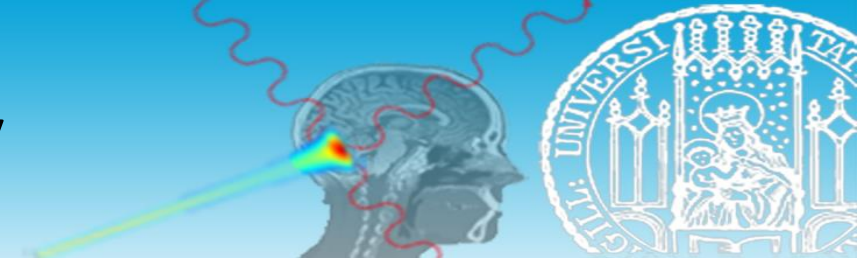


- The “3D-mode” is defined for **direct** and **oblique sinograms**

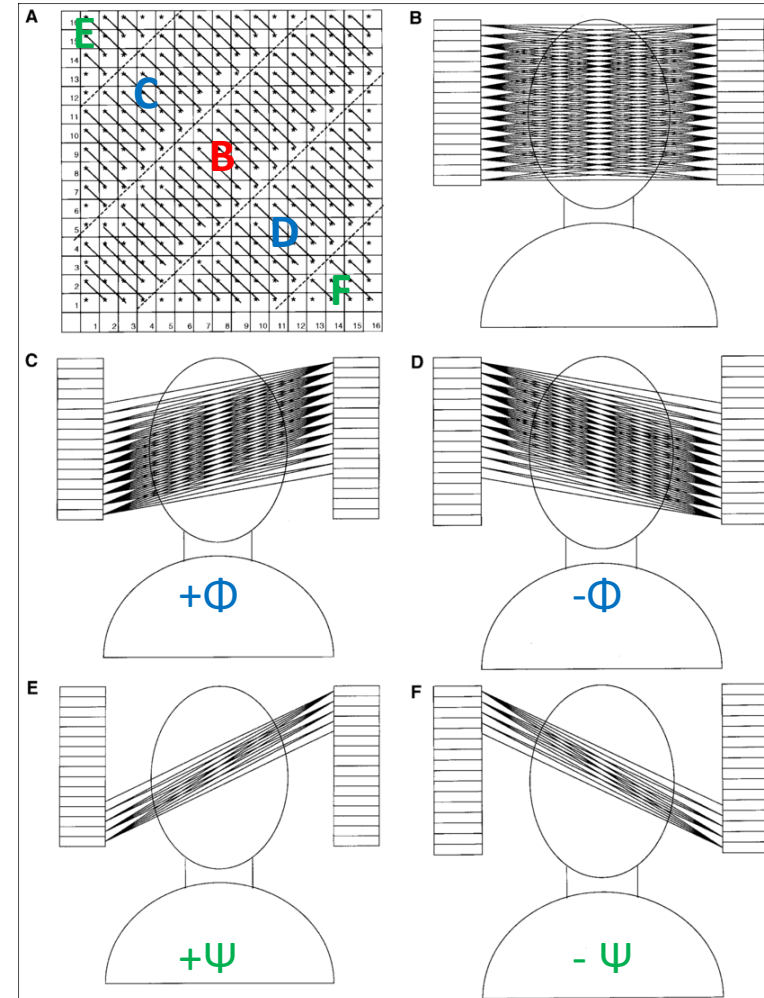
- 3D tomographic image reconstruction



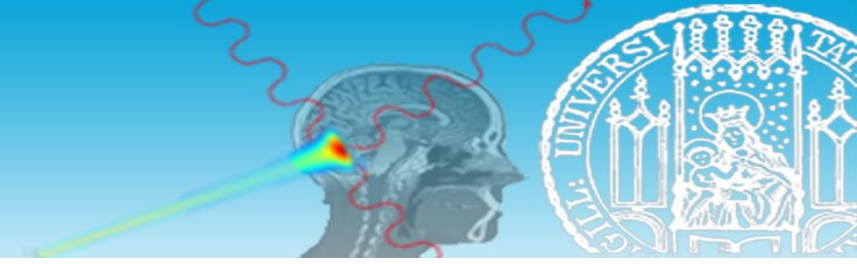
The imaging system geometry



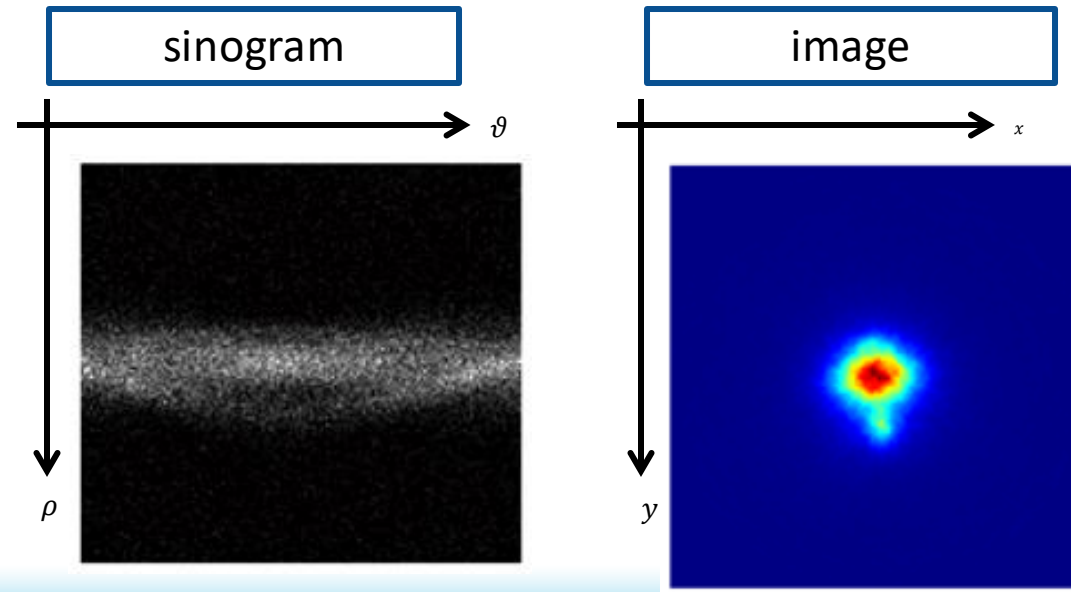
- A → example of 3D-mode **michelogram**
- B → direct sinograms and averaged oblique sinograms ($\Phi=0$)
- C, D → oblique sinograms, $+\Phi$ and $-\Phi$
- E, F → oblique sinograms, $+\Psi$ and $-\Psi$



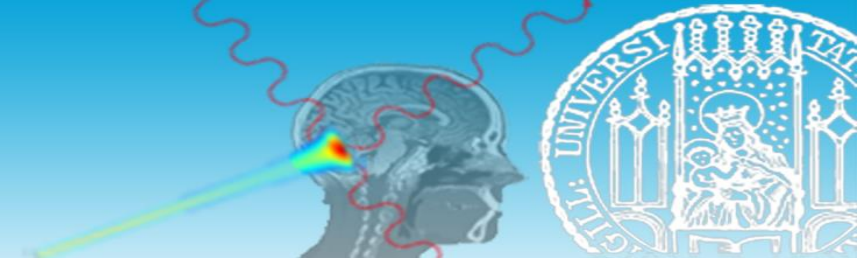
Poisson noise in PET imaging



- The observations in PET imaging are the measurements of the annihilation photons in coincidence, subsequent to radioactive decay
 - “Indirect” observation of the cause
- Poisson statistics describe random, independent events that occur at a fixed mean rate λ , as radioactive decay
 - Each **projection** is a **Poisson variable**
 - The sinogram is intrinsically affected by Poisson noise



Poisson noise in PET imaging



- Numerical image reconstruction aims at finding the image that satisfies:

$$f_{\min} = \operatorname{argmin}_{f_j} F(\bar{g}_i, g_i)$$

$$\bar{g}_i = \sum_{j=1}^J a_{ij} f_j$$

Model

$$g_i = \bar{g}_i + \text{noise}$$

Measurement

- Numerical image reconstruction is based on the **Poisson probability model** for radioactive decay

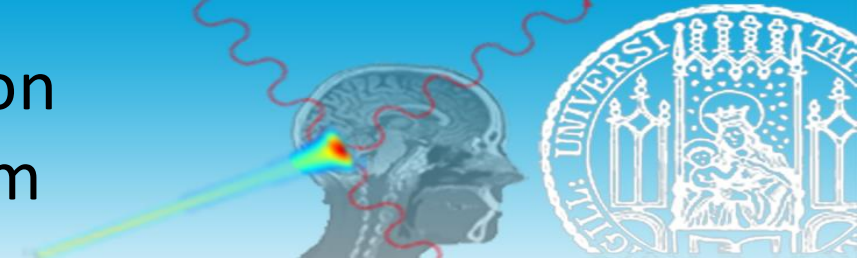
$$g_i \approx P(\bar{g}_i)$$

$$p(\bar{g}_i = g_i) = \frac{e^{-\bar{g}_i} \cdot \bar{g}_i^{g_i}}{g_i!}$$

The probability of observing k counts in a certain interval of time is defined by λ , or mean rate of counts, according to:

$$p(\lambda = k) = \frac{e^{-\lambda} \cdot \lambda^k}{k!}$$

Maximum Likelihood Expectation Maximization (MLEM) algorithm



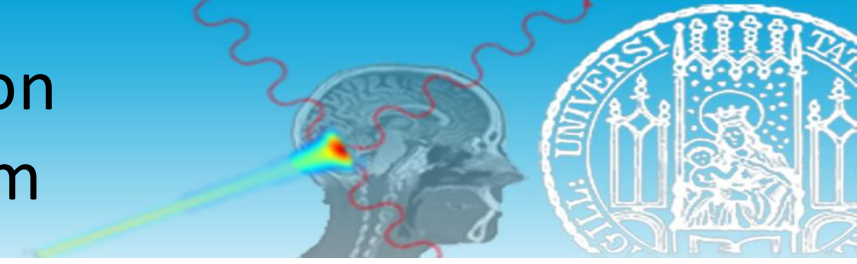
- The **update estimation** defines the tomographic reconstruction algorithm
 - The **Maximum Likelihood (ML)** approach is a method of estimating the parameters of a statistical model (i.e., the Poisson probability model for radioactive decay) given the observations (i.e., the number of counts detected along the LOR)

$$L(g, f) = p(\bar{g}_1 = g_1) \cdot p(\bar{g}_2 = g_2) \cdot \dots \cdot p(\bar{g}_I = g_I)$$

- The update estimation is based on the **Expectation Maximization (EM)** of the **likelihood function**, which express the probability to observe the measured projections g if the reconstructed image is f

$$\frac{\partial L(g, f)}{\partial f} = 0$$

Maximum Likelihood Expectation Maximization (MLEM) algorithm



- To simplify the maximization of L , the natural logarithm of the function is taken (the **log-likelihood**)
 - For $x > 0$ (the probability), both $y = x$ and $y = \ln(x)$ are minimized for $x \rightarrow 0$

$$\ln(L(g, f)) = \ln(p(\bar{g}_1 = g_1) \cdot p(\bar{g}_2 = g_2) \cdot \dots \cdot p(\bar{g}_I = g_I))$$

$$\ln\left(\prod_i \frac{e^{-\bar{g}_i \cdot \bar{g}_i^{g_i}}}{g_i!}\right) = \ln\left(\prod_i \frac{e^{-\sum_j a_{ij} f_j} \cdot (\sum_j a_{ij} f_j)^{g_i}}{g_i!}\right) = \ln \frac{e^{-\sum_i \sum_j a_{ij} f_j} \cdot \prod_i (\sum_j a_{ij} f_j)^{g_i}}{\prod_i g_i!} =$$

$$= -\sum_i \sum_j a_{ij} f_j + \sum_i g_i \left(\ln \sum_j a_{ij} f_j \right) - \sum_i \ln(g_i!)$$

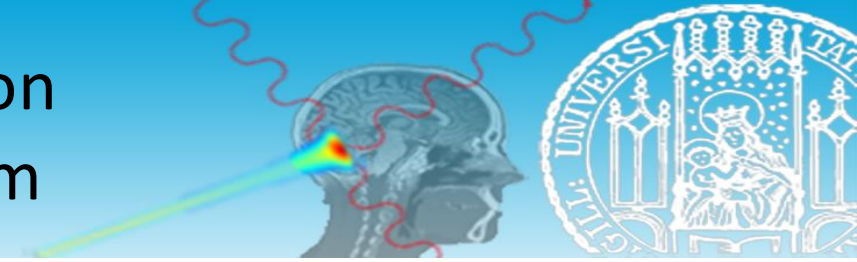
$$e^a e^b = e^{a+b}$$

$$\ln(a^b) = b \cdot \ln(a)$$

$$\ln(a \cdot b) = \ln(a) + \ln(b)$$

$$\ln(a/b) = \ln(a) - \ln(b)$$

Maximum Likelihood Expectation Maximization (MLEM) algorithm



- To simplify the maximization of L , the not observable variable x_{ij} (expectation or expected projection) is introduced

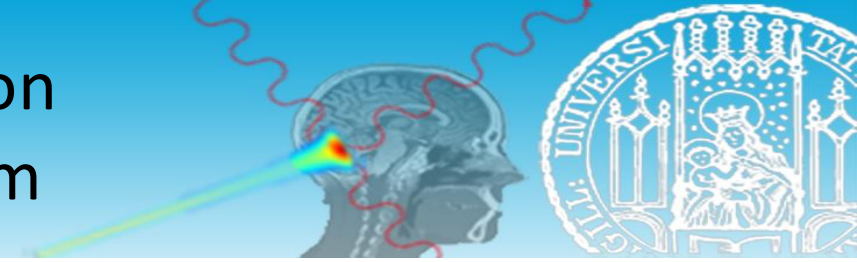
$$x_{ij} = \frac{a_{ij}f_j}{\sum_j a_{ij}f_j} g_i \quad \text{so that} \quad g_i = \sum_j a_{ij}f_j = \sum_j x_{ij}$$

- x_{ij} expresses the number of counts detected along the LOR i and emitted from the pixel j
- The projection is expressed in function of x_{ij}

$$\begin{aligned} \ln(L(g, f)) &= -\sum_i \sum_j a_{ij}f_j + \sum_i g_i \left(\ln \sum_j a_{ij}f_j \right) - \sum_i \ln(g_i!) = \\ &= \sum_i \left(\sum_j (-a_{ij}f_j + x_{ij} \ln(a_{ij}f_j) - \ln(x_{ij}!)) \right) = \\ &= \sum_i \sum_j (-a_{ij}f_j + x_{ij} \ln(a_{ij}f_j)) \end{aligned}$$

- The term independent by f_j is deleted from the maximization

Maximum Likelihood Expectation Maximization (MLEM) algorithm



Step by step

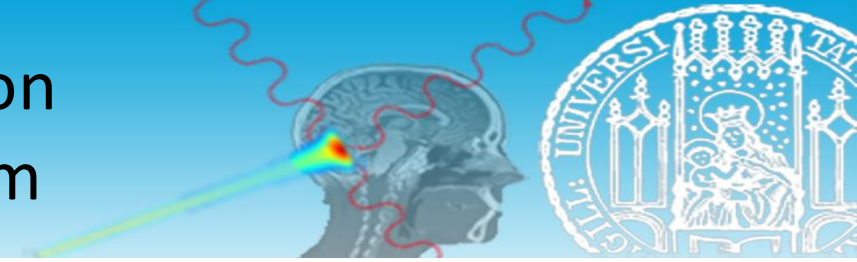
$$\ln(L(g, f)) = - \sum_i \sum_j a_{ij} f_j + \sum_i g_i \left(\ln \sum_j a_{ij} f_j \right) - \sum_i \ln(g_i!) =$$

$$\ln(L(g, f)) = - \sum_i \sum_j a_{ij} f_j + \sum_i \sum_j x_{ij} \left(\ln \sum_j a_{ij} f_j \right) - \sum_i \ln(g_i!) =$$

$$\ln(L(g, f)) = - \sum_i \sum_j a_{ij} f_j + \sum_i \sum_j (x_{ij} \ln(a_{ij} f_j)) - \sum_i \ln \left(\sum_j x_{ij}! \right) =$$

$$= \sum_i \sum_j (-a_{ij} f_j + x_{ij} \ln(a_{ij} f_j) - \ln(x_{ij}!)) =$$

Maximum Likelihood Expectation Maximization (MLEM) algorithm



- The maximization of L is based on the annulling of the first derivative

$$\frac{\partial L(g, f)}{\partial f} = \frac{\partial L(x, f)}{\partial f} = \sum_i \sum_j \left(-a_{ij} + \frac{x_{ij}}{f_j} \right) = 0$$

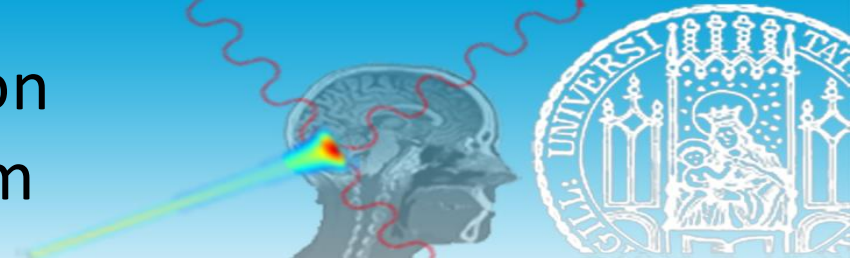
$$\frac{\partial L(g, f_j)}{\partial f_j} = \frac{\partial L(x, f_j)}{\partial f_j} = \sum_i \left(-a_{ij} + \frac{x_{ij}}{f_j} \right) \rightarrow \sum_i \left(-a_{ij} + \frac{x_{ij}}{f_j^{n+1}} \right) = 0$$

$$x_{ij} = \frac{a_{ij} f_j}{\sum_j a_{ij} f_j} g_i \rightarrow \frac{a_{ij} f_j^n}{\sum_j a_{ij} f_j^n} g_i$$

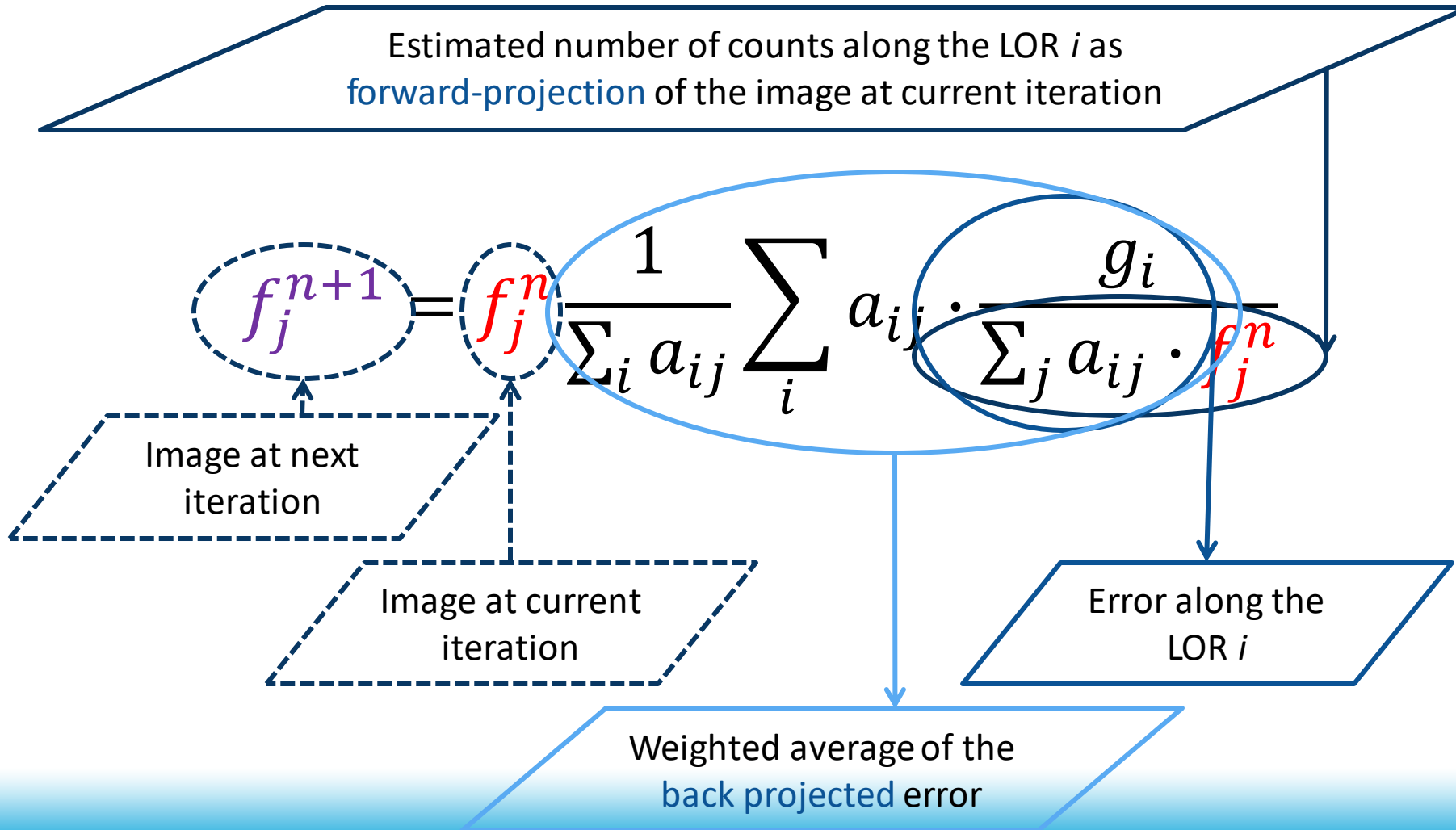
- The updating formula of the **ML-EM algorithm** is derived accordingly

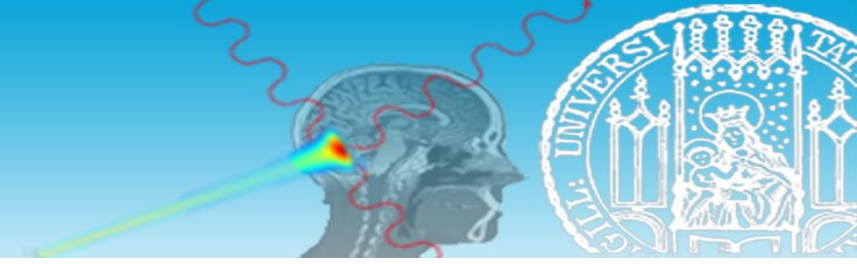
$$f_j^{n+1} = f_j^n \frac{1}{\sum_i a_{ij}} \sum_i \frac{a_{ij} g_i}{\sum_j a_{ij} f_j^n}$$

Maximum Likelihood Expectation Maximization (MLEM) algorithm



- Interpretation of the **multiplicative updating formula** of the ML-EM algorithm





- The noise on the projections makes the tomographic image reconstruction in PET imaging an **ill-posed inverse problem**
- The iterations of the ML-EM algorithm must be stopped before image convergence due to noise break-up

FBP, different windowing of the Ramp filter

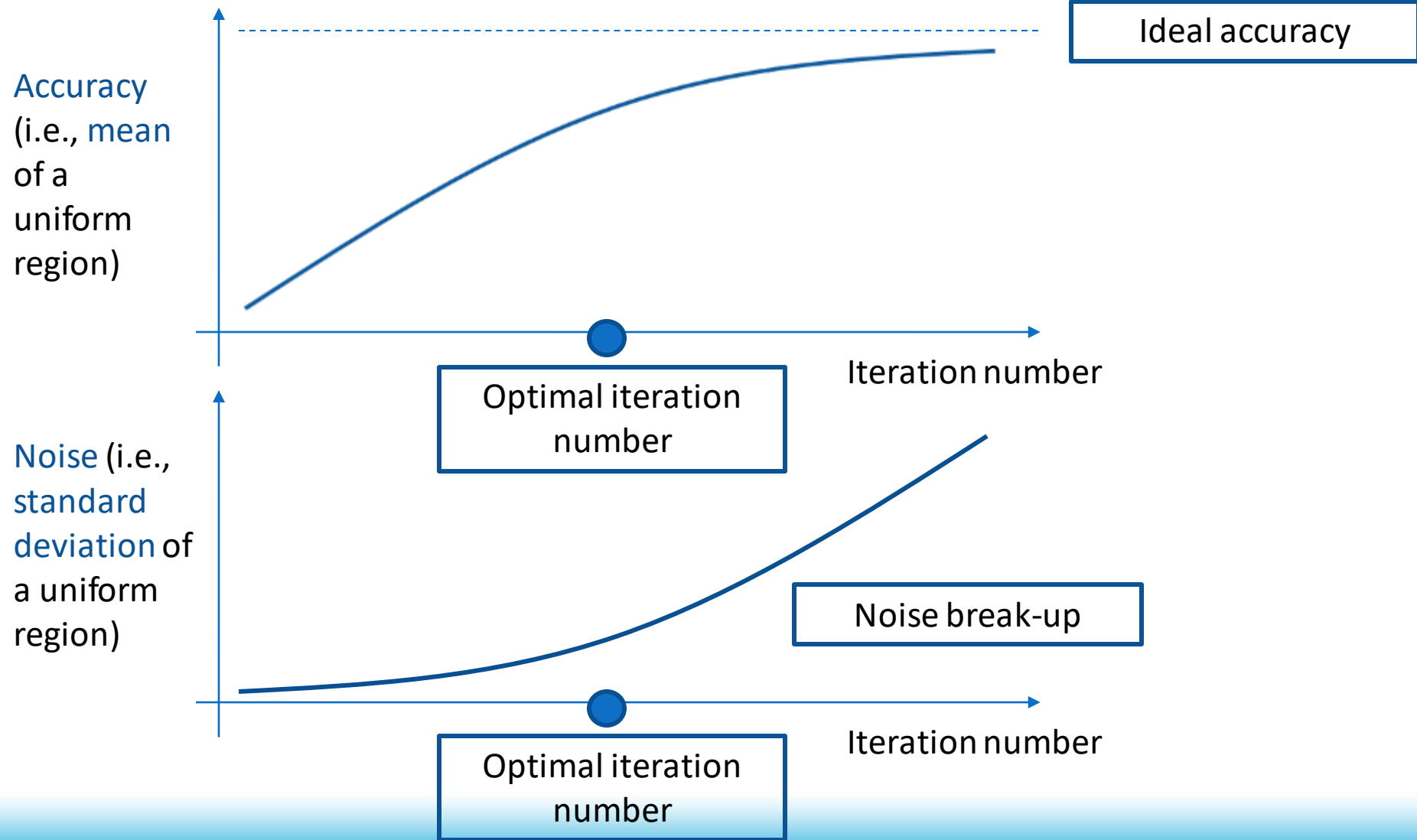
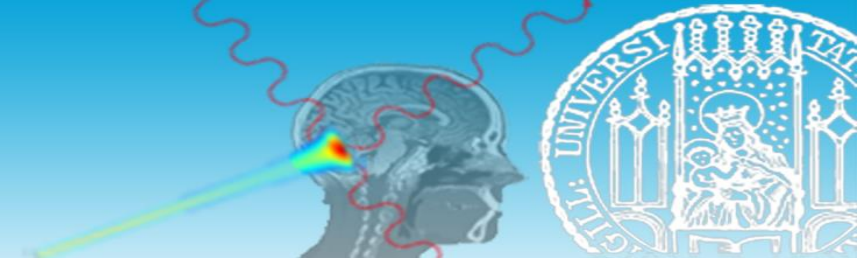
ML-EM, different number of iterations

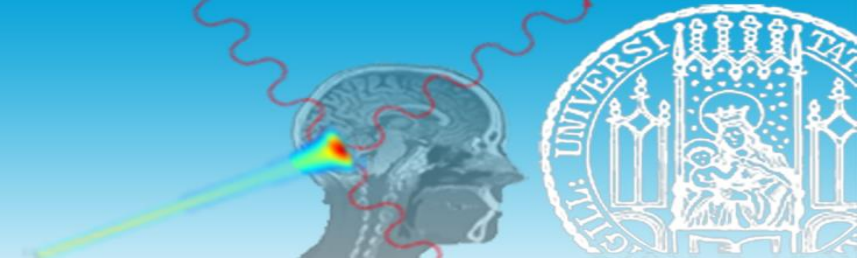


Adapted from Wieczorek, Herfried. "The image quality of FBP and MLEM reconstruction". *Physics in Medicine and Biology*, 55.11 (2010).

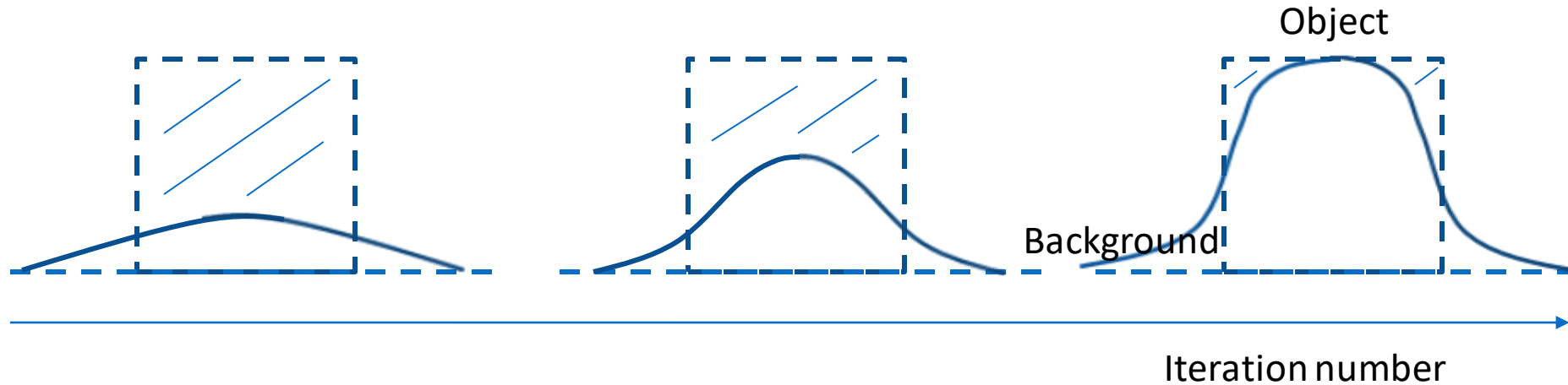
- This translates into a **trade-off between noise and accuracy (spatial resolution)** which affects clinical applications


Outlook

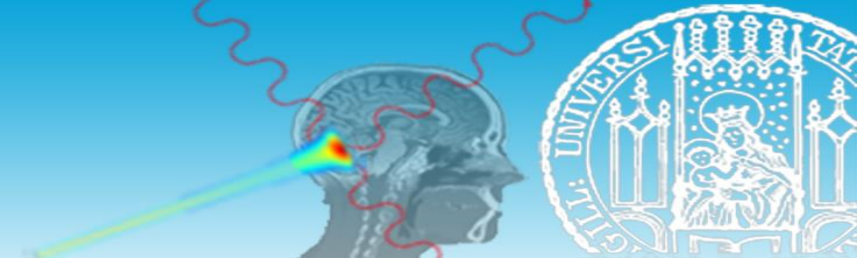




- The **accuracy** embeds information about **spatial resolution**



- An increase in **accuracy** (i.e., **mean** of a uniform region) corresponds to an increase of **spatial resolution** (i.e., **full width at half maximum**)
- The missing area/volume () of the object is known as “partial volume” and it is one of the most important limitations in quantitative PET imaging (i.e., quantification of tumor uptake)



- Numerical image reconstruction makes use of the **discrete form of the Radon Transform** (the sinogram)
- Numerical image reconstruction algorithms are simply enabled by the **modeling of the imaging system geometry in a system matrix**
- The choice of analytical or numerical reconstruction algorithms depends on the specific application in terms of geometry of the projection lines, angular coverage and angular sampling (i.e. **geometrical constraints**), noise level on the projections (i.e. **dosimetric constraints**)
 - If the continuity hypothesis of the image and the sinogram is matched and the noise level is low, **analytical image reconstruction algorithms** can be considered
 - Otherwise, more flexible (but more expensive under a computational point of view) **numerical image reconstruction algorithms** are preferred