Introductory course

Standard Model of Particle Physics

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Chapter 1

Introduction

1.1 Particles

The Standard Model (SM) of elementary particle interactions, also known as the Glashow-Weinberg-Salam model, is a gauge theory that describes the strong, weak and electromagnetic interactions of elementary particles. To begin, we make a brief overview of these particles and the main features of their interactions, which the model has to account for.

Ordinary matter is constituted by what is known as the first generation, or first family, of fermions. They are listed in the following table, along with their charges (in units of the electron charge) and their masses (in MeV)

		Q (e)	M (MeV)
Quarks	u	2/3	$\sim 1 - 10$
	d	-1/3	$\sim 1 - 10$
Leptons	ν_e	0	$\leq 10^{-6}$
	e	-1	0.5

All these are spin 1/2 fermions. Quarks have the so-called *color charge*, i.e. they can feel the strong force, while leptons are neutral under it. Besides these particles that make up ordinary matter: protons and neutrons built from quarks, and electrons - and neutrinos produced by weak interactions - in particle accelerators and high energy astrophysics processes more particles have been found, adding up to two more families. These are replicas of the first one, their members having the same quantum numbers, and differ only in their masses. Mass increases with family number for all particles, with the possible exceptions of neutrinos, whose family hierarchy we do not know yet. The previous table becomes

1st family

	Q (e)	M (MeV)
u	2/3	$\sim$ 1- 10
d	-1/3	$\sim$ 1 -10
ν_e	0	$\leq 10^{-6}$
e	-1	0.5

2nd family

	Q (e)	M (MeV)
c	2/3	1500
s	-1/3	$\sim$100
ν_μ	0	$\leq 10^{-6}$
μ	-1	106

3rd family

	Q (e)	M (MeV)
t	2/3	174300
b	-1/3	4500
ν_τ	0	$\leq 10^{-6}$
τ	-1	1784

1.2 Interactions

Particles in the three families have four types of interactions: gravitational, strong, weak and electromagnetic. In the Standard Model, gravitational interaction is too feeble to be able play any role. Let us make a brief review of the main features of the other three.

Electromagnetic Interaction

The quantum theory of electromagnetic interactions, the Quantum Electro-Dynamics (QED)is is based on a $U(1)$ gauge group. The interaction is mediated by a spin 1 gauge field, the photon. Among the SM particles, only neutrinos are immune to electromagnetic interactions.

Strong Interaction

Also called color interaction, it is felt by quarks only. Strong forces produce bounded states of three quarks (baryons, such as protons and neutrons) or a quark-antiquark pair (mesons). Its most relevant feature is *asymptotic freedom*, which we shall not discuss here. Suffice it to say that as a consequence of this, free colored states cannot be found in nature, only bounded, color-neutral ("white") states are allowed.

Strong interactions are described by the Yang-Mills gauge theory, based on the group $SU(3)_C$, called Quantum Chromo-Dynamics (QCD). Since the adjoint representation of $SU(3)$ has dimension eight, this implies the existence of eight gauge fields, commonly called "gluons". Gluons are neutral under both electromagnetic and weak interactions.

Weak Interaction

Weak interaction is responsible for the well-known β decay process

$$n \to p + e + \overline{\nu_e}$$

which can be represented, in terms of the elementary components of neutrons and protons, as in the diagram of Fig. 1.1

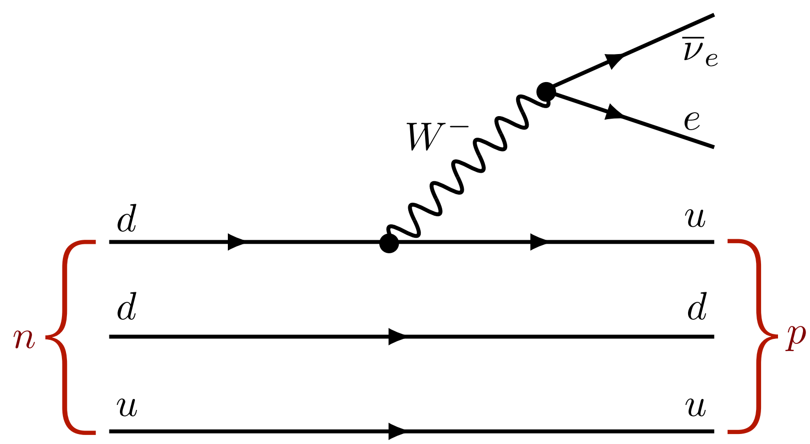


Figure 1.1: β decay

What we would like to have is a Yang-Mills theory with at least one gauge boson that can play the role of W in the diagram, mediating weak interaction just as the photon mediates the electromagnetic one. In the next section we will list the phenomenological features expected for such an interaction. For the time being, form the diagram one can see that the postulated boson W should have electromagnetic charge, if it is to be conserved on the vertices. That means that the generators of the symmetry group of the weak interactions, T_L, are not going too commute with electromagnetic charge

$$[T_L, Q] \neq 0 \tag{1.1}$$

On the other hand, as we shall see, the gauge field W turn out to be massive. This means that the symmetry beneath the weak interaction is broken in nature, in fact there is no such symmetry.

However, a first glimpse of the existence of a symmetry group associated with the weak interaction is the conservation of the so-called *isospin*. This quantum number is associated with a $SU(2)_I$ symmetry defined so that protons and neutrons are the components of a doublet

$$\begin{pmatrix} p \\ n \end{pmatrix} \tag{1.2}$$

Isospin is conserved in the nuclear interactions, those that involve only the strong force. In other words, the generators of $SU(2)_I$, T_I, satisfy

$$[T_I, H_s] = 0, \quad [T_I, H_{em}] \neq 0 \tag{1.3}$$

Where H_s is the effective Hamiltonian of nuclear interactions, H_{em} the electromagnetic one. As can be seen, the isospin group is not a symmetry of the fundamental interactions, only of a part of them, however, it gives us a first motivation for considering the group $SU(2)$ in building the theory.

The SM that we wish to build has to describe the above mentioned interactions between fermions of the three families. In order to do that, we will have to find the symmetry group of the theory, comprising the Poincaré group (Λ) and the internal symmetry groups (G_i):

$$G = \Lambda \times G_i \tag{1.4}$$

Let us see what are the building blocks of this description.

1.3 Building blocks

All the particles in the table have to be described by objects transforming under irreducible representations of the group G. We still do not know which group is this, but we do know that it must contain the Poincaré group, therefore our fields must transform as irreducible representations of the Lorentz group. We have to describe two kinds of particles¿ fermions and gauge bosons.

Fermions

Fermions are described with spinorial representations of the Lorentz group. That is, a representation where the group generators can be written as

$$\Sigma^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu] \tag{1.5}$$

where the γ are 4 matrices of dimension 4, which follow a Dirac algebra [1]

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{1.6}$$

In this way, the elements of the Lorentz group can be written in terms of the six parameters $\theta_{[\mu\nu]}$ (3 boosts and 3 rotations), as

$$\Lambda = e^{i\theta_{\mu\nu}\Sigma^{\mu\nu}} \tag{1.7}$$

Fermions are then given in terms of *four* fields, and transform under Lorentz group as

$$\psi' = \Lambda\psi \tag{1.8}$$

A free fermion with a mass m follows Dirac's equation

$$(\imath\gamma^\mu\partial_\mu - m)\psi \equiv (i \not\partial - m)\psi = 0 \tag{1.9}$$

However, this 4-dimensional representation of the Lorentz group is not irreducible. This can be easily seen by choosing a basis in group space where the generators are in block-diagonal form

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{1.10}$$

In this basis we have

$$\gamma_5 \equiv i\gamma^1\gamma^2\gamma^3\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.11}$$

(with σ_i the Pauli matrices), and we can write the group elements in terms of $\phi_i \equiv \theta_{0i}$, $\theta_i = \epsilon_{ijk}\theta_{jk}$:

Boosts:

$$e^{i\theta_{0i}\Sigma^{0i}} = \begin{pmatrix} e^{-\vec{\phi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{\vec{\phi}\cdot\vec{\sigma}/2} \end{pmatrix} \tag{1.12}$$

Rotations:

$$e^{i\theta_{ij}\Sigma^{ij}} = \begin{pmatrix} e^{-i\vec{\theta}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{-i\vec{\theta}\cdot\vec{\sigma}/2} \end{pmatrix} \tag{1.13}$$

[1]Here and in the following, greek subindices run from 0 to 3, latin from 1 to 3, and we shall use $g_{\mu,\nu} = diag(1, -1, -1, -1)$

It can be seem that the 4-dimensional representation decomposes into two two-dimensional *irreducible* representations. The projectors over the corresponding subspaces are

$$L \equiv \left(\frac{1-\gamma_5}{2}\right), \quad R \equiv \left(\frac{1+\gamma_5}{2}\right) \tag{1.14}$$

This way, we define the spinors

$$\psi_L \equiv L\psi, \quad \psi_R \equiv R\psi \tag{1.15}$$

Notice that each has in fact two components, although they are written as 4-dimensional vectors for convenience. They transform under the irreducible representations

$$\Lambda_R = R\Lambda = \begin{pmatrix} e^{-i(\vec{\theta}-i\vec{\phi})\cdot\vec{\sigma}/2} & 0 \\ 0 & 0 \end{pmatrix} \quad \Lambda_L = L\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & e^{-i(\vec{\theta}+i\vec{\phi})\cdot\vec{\sigma}/2} \end{pmatrix} \tag{1.16}$$

Notice also that $(\Lambda_L)^\dagger = (\Lambda_R)^{-1}$, that is, $\psi^\dagger\psi$ is not Lorentz invariant. It is sometimes convenient to write Dirac spinors in terms of two-component Weyl spinors:

$$\psi = \begin{pmatrix} \varphi_R \\ \varphi_L \end{pmatrix} \tag{1.17}$$

Parity

these two-component spinors are related by a parity transformation, defined as the one that changes the sign of the spatial components of the coordinates:

$$\psi_P(-\vec{x}, t) = P\psi(\vec{x}, t) \tag{1.18}$$

Using Dirac's equation, it is straightforward to show that the parity operator is

$$P = \gamma^0 \tag{1.19}$$

That is, under parity:

$$\varphi_L \to \varphi_R \tag{1.20}$$

Charge conjugation

Dirac's equation for a particle coupled to a $U(1)$ gauge field is

$$(i\gamma^\mu\partial_\mu + e\gamma^\mu A_\mu - m)\psi = 0 \tag{1.21}$$

$\overline{\psi}\psi =$	$\overline{\psi}_R\psi_L + \overline{\psi}_L\psi_R$	Dirac mass
$\psi^T C\psi =$	$\psi_L^T C\psi_L + \psi_R^T C\psi_R + h.c.$	Majorana mass
$\overline{\psi}\gamma^\mu\partial_\mu\psi =$	$\overline{\psi}_L\gamma^\mu\partial_\mu\psi_L + \overline{\psi}_R\gamma^\mu\partial_\mu\psi_R$	kinetic term

Table 1.1: Lorentz invariants with fermions

The *charge conjugated* spinor ψ^c satisfies:

$$(i\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu - m)\psi^c = 0 \tag{1.22}$$

Or

$$\psi^c \equiv C\overline{\psi}^T = i\gamma^2\gamma^0\overline{\psi}^T = i\gamma^2\psi^* \tag{1.23}$$

Therefore under charge conjugation

$$\varphi_L \to -i\sigma_2\varphi_R^* \quad \varphi_R \to i\sigma_2\varphi_L^* \tag{1.24}$$

Helicity

Dirac' s equation mixes the L and R components of spinors, in momentum space we have

$$(\not{p} - m)\psi = 0 \quad \Rightarrow \quad \not{p}\psi_R = m\psi_L \tag{1.25}$$

For a massless particle, or in the approximation $p \gg m$:

$$\not{p}\psi_L = \begin{pmatrix} 0 & p_0 - \vec{p}.\vec{\sigma} \\ p_0 + \vec{p}.\vec{\sigma} & 0 \end{pmatrix} \begin{pmatrix} \varphi_R \\ 0 \end{pmatrix} = 0 \tag{1.26}$$

And an analogous relation for ψ_L. We see that the Weyl spinors φ_L, φ_R are eigenstates of the *helicity operator*, $\hat{h}$:

$$\hat{h}\varphi_L \equiv \frac{\vec{p}.\vec{\sigma}}{2p_0}\varphi_L = \frac{1}{2}\varphi_L \tag{1.27}$$

$$\hat{h}\varphi_R \equiv \frac{\vec{p}.\vec{\sigma}}{2p_0}\varphi_R = -\frac{1}{2}\varphi_R \tag{1.28}$$

A "Dirac particle" is therefore built up with two helicity states, and one an write a Dirac mass term for it as in table 1.1.

If we have a massless particle, with only one helicity state available, we can write a Majorana spinor, by imposing

$$\varphi_R = -i\sigma_2\varphi_L^* \tag{1.29}$$

It is easily shown that both sides of the equation transform under the representation Λ_R. Such a particle allows for a Majorana mass term:

$$\psi^T C\psi + h.c. = -\varphi_L^T i\sigma_2\varphi_L + h.c. \tag{1.30}$$

However, notice that this mass term, involving only one helicity state, is not invariant under any non trivial symmetry group, such as the ones that we will need for the internal symmetries. Particles with a Majorana mass term must be totally neutral.

Gauge Bosons

In a Yang-Mills theory, derivatives of fields transforming under the fundamental representation are replaced by the covariant derivative

$$\partial_\mu\Phi \to (\partial_\mu - igA_\mu)\Phi \tag{1.31}$$

where g is the gauge coupling constant and A_μ is the gauge field, written as a linear combination of the group generators in the fundamental representation, T_a:

$$A_\mu = A_\mu^a T_a \tag{1.32}$$

Under a gauge transformation:

$$\Phi \to U\Phi \equiv e^{i\alpha_a(x)T^a}\Phi \tag{1.33}$$

$$A_\mu \to UA_\mu U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger \tag{1.34}$$

If α is an infinitesimal parameter:

$$A_\mu^a \to A_\mu^a - \frac{1}{g}\partial_\mu\alpha^a + f^{abc}\alpha^b A_\mu^c \tag{1.35}$$

where f^{abc} are the group's coupling constants. In other words, the gauge fields suffer a *gauge* transformation and also a *group* transformation, under the adjoint representation. The kinetic term for gauge fields is

$$F^a_{\mu\nu}F^{\mu\nu a}, \quad F^a_{\mu\nu}T^a = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^a_\mu A^b_\nu T^c \tag{1.36}$$

Gauge invariance forbids any mass term

$$A_\mu A^\mu \tag{1.37}$$

So that Yang-Mills theories can describe interactions mediated by massless fields only.

QED

As a first approximation to the theory we wish to build, let us for the moment ignore strong and weak interactions. The QED Lagrangian is

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \sum_f (i\overline{\psi}_f\gamma^\mu D_\mu\psi_f - m\overline{\psi}_f\psi_f) \tag{1.38}$$

$$D_\mu = \partial_\mu - ieq_f A_\mu \tag{1.39}$$

where the subindex f labels the fermions and q_f are their electromagnetic charges. The electromagnetic current is

$$J^{em}_\mu = \overline{\psi}\gamma_\mu\psi \tag{1.40}$$

and charge is conserved.

For the first generation of fermions we will have

$$\begin{aligned}\mathcal{L}_{QED} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\overline{e}(\gamma^\mu\partial_\mu + ieA_\mu)e + i\overline{u}(\gamma^\mu\partial_\mu - i\frac{2}{3}eA_\mu)u \\ &+ i\overline{d}(\gamma^\mu\partial_\mu + i\frac{1}{3}eA_\mu)d + i\overline{\nu}_e\gamma^\mu\partial_\mu\nu_e + (\text{term. de masa})\end{aligned} \tag{1.41}$$

It will be often convenient to write the interaction terms with the photon (and other gauge fields) as i.e.:

$$\overline{e}\gamma^\mu A_\mu e = A_\mu J^\mu_e \tag{1.42}$$

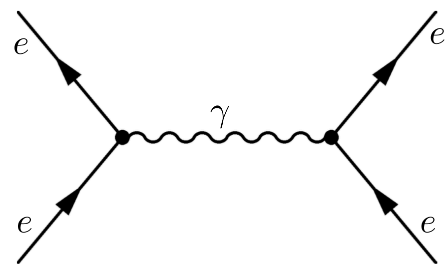


Figure 1.2: Electron-electron scattering

Box 1.1: Electromagnetic potential

We wish to find the electromagnetic potential between two particles with charge q. Consider for example electron-electron dispersion as in the diagram of Fig.1.3 It will give an effective interaction

$$q^2\, \bar{e}\gamma^\mu e \left(\frac{-ig_{\mu\nu}}{k^2}\right) \tag{1.43}$$

In the non-relativistic limit, when $m_e \gg |\vec{p}|, |\vec{p}'|$, *it can be shown that*

$$\bar{e}\gamma^\mu e \simeq \bar{e}\gamma^0 e = e^\dagger e \tag{1.44}$$

Therefore:

$$q\,(\bar{e}\gamma^\mu e) \simeq q\, J_0 = Q \tag{1.45}$$

Going to coordinate space, 1.43 gives the interaction potential in terms of the charge density Q:

$$V(\vec{r}) = -iQ^2 \int d^3k \frac{1}{k^2} e^{i\vec{k}\cdot\vec{r}} = -2\pi i Q^2 \int k^2 dk \int \frac{1}{k^2} e^{ir\cos\theta} \tag{1.46}$$

Performing the angular integration

$$V(r) = -\frac{2\pi i}{r} Q^2 \int kdk \frac{1}{k^2}(e^{ikr} - e^{-ikr}) \tag{1.47}$$

Now, suppose the gauge field A_μ was massive. Changing

$$k^2 \to k^2 + m^2 \tag{1.48}$$

the integral in 1.47 becomes one with poles in $\pm im$, so that

$$\int_{-\infty}^{\infty} e^{ikr} \frac{kdk}{k^2 + m^2} = \pi i e^{-mr} \tag{1.49}$$

and the same result is obtained for the other integral. Then

$$V(r) = 4\pi^2 \frac{Q^2}{r} e^{-mr} \tag{1.50}$$

We see that for a massless gauge field, the usual potential is obtained. However, if somehow we allow for $m \neq 0$, we obtain a potential that decreases very fast with distance: a short range force.

Up to now, in order to build the SM we need

1. **symmetry group**: $G =$ Poincaré + G_i
2. **fermions**: irreducible representations of G; chiral Lorentz spinors ; fundamental representations of the groups in G_i
3. **gauge bosons**: Lorentz vectors, adjoint representations of gropus in G_i. Massless.

The group G_i

What would be the appropriate internal symmetry group G_i? We know it must contain

$$SU(3)_C - \text{strong interaction}$$

Furthermore, we expect to have

$$U(1)_{em}, \quad SU(2)_I$$

The strong interaction group is an exact symmetry of nature, as is electromagnetism. But in addition, the strong interaction Hamiltonian commutes with the electromagnetic and isospin ones. We can therefore up to now expect

$$G_i = SU(3)_c \times G_{sm} \tag{1.51}$$

Where G_{sm} is the group that will describe eventually electromagnetism and weak interactions. In fact $SU(3)$ is an exact symmetry and strong interactions are simply described as a Yang-Mills theory based in $SU(3)$, with the three families of quarks as a fermionic content. The study of the Standard Model usually does not involve QCD. Here we will only give a brief review of the most important features of strong interactions:

- The symmetry is exact, the 8 gluons are massless. Furthermore, only neutral bounded states can be observed.
- The $SU(3)_c$ $SU(3)_c$ generators are the 8 Gell-Mann matrices, λ_a, with a=1..8. Therefore the gluons are written

$$G_\mu = G_\mu^a \lambda_a$$

- Only quarks transform non trivially under $SU(3)_c$, and they are in the fundamental representation:

$$q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

- Currents are not neutral in color, they are written

$$J_\mu^{ij} = \overline{q}^i \gamma_\mu q^j$$

- The QCD Lagrangian for the first family looks like

$$\mathcal{L}_{QCD} = -\frac{1}{4} G_{\mu\nu}^a G_a^{\mu\nu} + \overline{u}_i \gamma^\mu (\partial_\mu \delta_{ij} + igG_\mu^a \lambda_{ij}^a) u_j \tag{1.52}$$

$$+ \overline{d}_i \gamma^\mu (\partial_\mu \delta_{ij} + igG_\mu^a \lambda_{ij}^a) d_j + m_u \overline{u}_i u_i + m_d \overline{d}_i d_i \tag{1.53}$$

Notice that no Majorana mass terms are allowed, and that it preserves P and C.

In the next chapter we will present some basics of weak interaction phenomenology, as an attempt to determine what group G_{sm} is suitable.

Exercise 1
Find the Charge conjugation and Parity operators.

Exercise 2
Show that the terms listed in table 1.1 are indeed Lorentz invariants, and find how they transform under C, P y CP.

Exercise 3
Using Dirac representation for the γ^μ matrices, take the non/relativistic limit and find (1.44).

Exercise 4
Fill in the intermediate steps leading to (1.50).

Chapter 2

Weak interactions

β decay can be described with the Fermi effective Lagrangean, for protons and neutrons

$$\mathcal{L}_F = -\frac{G_F}{\sqrt{2}}[\overline{p}\gamma_\mu n][\overline{e}\gamma^\mu \nu_e] + h.c. = -\frac{G_F}{\sqrt{2}} J_\mu^\dagger J^\mu \tag{2.1}$$

where G_F, Fermi's constant, measures the strength of the interaction.

This effective Lagrangean must come from a more fundamental theory involving quarks and leptons. Later on, a series of experimental and theoretical findings allowed for a better description of the structure of this interaction. We will not give here a historical review, but rather give the result and some examples of processes that illustrate how it was found.

The effective current can be separated into leptonic J_ℓ^μ and hadronic J_h^μ parts:

$$J^\mu = J_\ell^\mu + J_h^\mu \tag{2.2}$$

con

$$J_\ell^\lambda = \overline{\nu}_e \gamma^\lambda (1 - \gamma_5) e \tag{2.3}$$

$$J_h^\lambda = \overline{u}\gamma^\lambda (1 - \gamma_5) d_\theta; \quad d_\theta = \cos(\theta_c) d + \sin(\theta_c) s \tag{2.4}$$

Notice: Hadronic current involves the s quark, the first member of the second family to be discovered, in the so called *strange* processes (hence the quark's name). The parameter θ_c, or Cabbibo angle, is an experimental result. But the most important observation is the presence of the projector $L = (1-\gamma_5)/2$, which implies a *maximal parity violation* in weak interactions. This is an experimental fact.

Box 2.1: $\not{P}$: W DECAY

A weak interaction boson W^-, a spin=1 particle, can decay into an electron and an antineutrino:

$$W^- \to e + \overline{\nu}_e \tag{2.5}$$

W^- is much heavier than the electron, $m_W \gg m_e$, so we can work in the approximation of the electron consisting of two helicity eigenstates:

$$e = \begin{pmatrix} e_R \\ e_L \end{pmatrix} \qquad \hat{h}\, e_{L(R)} = \pm\frac{1}{2} e_{L(R)} \tag{2.6}$$

Since the neutrino is massless, it will be similarly composed of pure helicity states.

We will demand that 2.5 conserves total angular momentum

$$J_{initial} = J_{final} \tag{2.7}$$

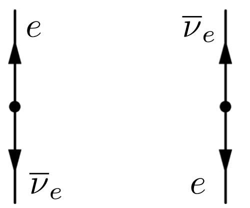
If we start with a polarized beam of W^-, we can choose the reference frame so that the boson is at rest and its spin $\vec{S}$ points in the z direction:

$$J^i_{initial} = S_z \delta^i_z = 1\delta^i_z \tag{2.8}$$

Furthermore, we must conserve momentum, so e and ν travel in opposite directions. There are two possibilities, as shown in figure 2

(a) *If e travels in the positive z direction:*

$$J_z^{final} = 1 = s_z^e + s_z^\nu = \frac{1}{2} + \frac{1}{2} \tag{2.9}$$



(a) (b)

Remembering that helicity is defined as the projection $\hat{p}.\vec{s}$ we have

$$\hat{h}(e) = \hat{p}.s_z^e = s_z^e = 1/2 \quad \hat{h}(\overline{\nu}) = \hat{p}.s_z^\nu = -s_z^\nu = -1/2 \tag{2.10}$$

In other words, this case involves left-handed states, e_L and ν_L.

(b) *If e travels in the negative z direction:*

$$\hat{h}(e) = \hat{p}.s_z^e = -s_z^e = -1/2 \quad \hat{h}(\overline{\nu}) = \hat{p}.s_z^\nu = s_z^\nu = 1/2 \tag{2.11}$$

In this case right-handed states e_R and ν_R participate.

In W^- decay, one measures the number of electrons coming off in both directions, through (a) or (b). If parity is conserved, one expects to find half the electron going up and half going down. What is observed is possibility (a), with a 99% probability. *In other words, only left-handed states interact with W^-.*

Box 2.2: Pion decay

Pions are mesons, made up of a quark and an antiquark:

$$\pi^- = \overline{u}d \tag{2.12}$$

with mass $m_\pi \sim 140MeV$ *and spin=0. They can in principle decay via weak interaction (with a long lifetime) via two channels:*

$$\text{(a)} \qquad \pi^- \to e + \overline{\nu}_e$$

$$\text{(b)} \qquad \pi^- \to \mu + \overline{\nu}_\mu$$

However, channel (a) *is not observed. As* $m_\pi \gg m_e$*, the electron can be considered massless, and the same reasoning as in the W decay can be applied. In the rest frame of* π *we will have*

$$J_{initial} = 0; \quad J_{final} = s_z^e + s_z^\nu = \pm 1 \tag{2.13}$$

depending on the direction the electron takes. The process cannot happen.

On the other hand, if we consider the muon, its mass is comparable with π*, the helicity states of* μ *are inseparable, producing effects of order* m_μ*:*

$$\Gamma_{\pi\to\mu} \propto m_\mu^2 \tag{2.14}$$

In fact, one observes precisely:

$$\frac{\Gamma_{\pi\to\mu}}{\Gamma_{\pi\to e}} \simeq \left(\frac{m_\mu}{m_e}\right)^2 \sim 10^4 \tag{2.15}$$

2.1 V-A theory

Since

$$\overline{\psi}\gamma_\mu\psi \tag{2.16}$$

transforms as a Lorentz vector, while

$$\overline{\psi}\gamma_\mu\gamma_5\psi \tag{2.17}$$

transforms as a pseudovector or axial vector (it changes sign under parity), the electroweak Lagrangean, with terms like

$$\overline{\psi}\gamma_\mu(1-\gamma_5)\psi \tag{2.18}$$

is usually called $V - A$ Lagrangean. It can be seen explicitly that it does not commute with parity

⇝ *Weak interactions violate parity maximally*

Let us try and write this Lagrangean as coming from a non-abelian gauge theory. We do not know what is the symmetry group, but for the time being we will include the bosons observed before the theory was constructed, the charged ones W^- y W^+. From the covariant derivative terms we will get an interaction

$$\mathcal{L}_{int} = g\overline{\nu}_e\gamma^\lambda(1-\gamma_5)eW^+_\lambda + g\overline{u}\gamma^\lambda(1-\gamma_5)d_\theta W^+_\lambda \tag{2.19}$$

We see that the currents J^λ_ℓ, J^λ_h are charged. Again, this means that, if T_a are the generators of the weak interactions group, we have

$$[T_a, Q_{em}] \neq 0 \tag{2.20}$$

This means that the symmetry group cannot be an external product $U(1)_{em} \times G_{weak}$

⇝ *Weak interactions involve charged currents*

In the (V-A) Lagrangean, we do not have just first family quarks, but rather the combination

$$d_\theta = cos(\theta_c)d + sin(\theta_c)s \tag{2.21}$$

However, the same is not true in the leptonic sector

⇝ *Weak interactions mix quark families, but not lepton families*

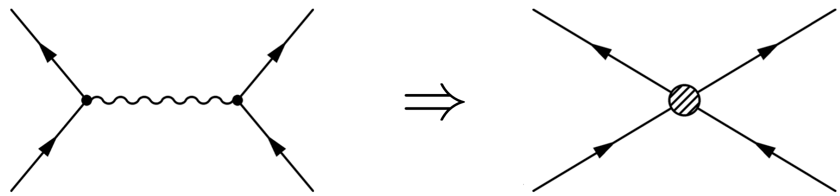


Figure 2.1: Fermi Lagrangean from a gauge theory

On the other hand, the Fermi Lagrangean indicates the existence of 4-fermion effective interactions. This is the same as saying that we have an interaction with a massive gauge bosons, as in figure 2.1 We get

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{M_W^2} \tag{2.22}$$

where g is the gauge coupling and M_W the gauge boson's mass. But a theory with massive gauge bosons is not renormalizable. If we write the Lagrangean for W

$$\mathcal{L}_W = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + M_W^2 W_\mu^\dagger W^\mu \tag{2.23}$$

we get a propagator

$$\Delta_{\mu\nu} = -\frac{g_{\mu\nu} - k_\mu k_\nu / M_W^2}{k^2 - M_W^2} \tag{2.24}$$

For large momenta, $k \to \infty$:

$$\Delta_{\mu\nu} \to (const).\frac{1}{M_W^2} \tag{2.25}$$

we get a non-renormalizable theory. This is not strange, since the mass term has destroyed the gauge invariance that protected renormalizability.

$\rightsquigarrow$ *Weak interactions have massive gauge bosons, which destroy gauge invariance and renormalizability.*

Anyway, suppose we are stubborn and insist in writing the theory based in a certain symmetry group G_{weak}. As we saw, only left-handed fermions will transform trivially under this group. Let's say they do it in the fundamental representation, and hat this group is $SU(2)$ (like isospin). We will have three gauge bosons

$$(W_\mu^1, W_\mu^2, W_\mu^3) \tag{2.26}$$

If we define

$$\ell_L \equiv \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \tag{2.27}$$

from the covariant derivative we will get a term

$$\overline{\ell}_L^i \gamma^\mu (W_\mu^a T_{ij}^a) \ell_L^j \tag{2.28}$$

and T^a will be given by the Pauli matrices, i.e.

$$W_\mu^a T_a = W^+ \sigma^+ + W^- \sigma^- + W^3 \sigma^3 \tag{2.29}$$

where we have set $\sigma^\pm = \sigma^1 \pm i\sigma^2$ y $W_\mu^\pm = (W_\mu^1 \pm iW_\mu^2)/\sqrt{2}$. This term will produce an interaction

$$\begin{pmatrix} \overline{\nu}_L & \overline{e}_L \end{pmatrix} \gamma^\mu W_\mu^+ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} = \overline{\nu}_L \gamma^\mu e_L W_\mu^+ \tag{2.30}$$

which is precisely what we want. At least in the leptonic sector, where families are not mixed, $SU(2)$ is a good candidate for the weak group. However, we have placed inside the doublet ℓ_L two particles of different mass, $m_\nu \neq m_e$. This is equivalent to saying that the group generators do not commute with the

$$[H, T_a] \neq 0 \tag{2.31}$$

In fact, they do not commute with the electromagnetic one either (ν and e charges being different), only with the color Hamiltonian.

$\rightsquigarrow$ *The symmetry group of weak interactions does not commute with the Hamiltonian, the symmetry is broken .*

2.2 First try

We have tried to build a theory of weak interactions analogous to QED and QCD based in some group G_{weak}, but we have found

1. only ψ_L are involved in weak interactions
2. weak currents are charged
3. currents mix quark families, but not leptons
4. gauge bosons are massive
5. $[G_{weak}, H] \neq 0$: there is no such symmetry

All these features, as we shall see, can be explained if G_{weak} is spontaneously broken. Before that, let us make a first approximation to the building of the theory, ignoring for the moments the problems such as the gauge boson's mass and considering only the first family

The first candidate, suggested by the $V - A$ structure, is $SU(2)$. Extending the results above for the leptonic currents, let us say that all left-handed spinors transform as $SU(2)$ doublets. We of course group colored particles separately from white ones, so our choice is:

$$\ell_L \equiv \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \ ; \quad q_L \equiv \begin{pmatrix} u_L \\ d_L \end{pmatrix} \tag{2.32}$$

The right-handed particles we choose to be $SU(2)$ singlets

$$e_R \ ; \quad u_R \ ; \quad d_R \tag{2.33}$$

Right-handed neutrinos are not present. We have three gauge bosons in the adjoint of $SU(2)$:

$$W_\mu^+ \ ; \quad W_\mu^- \ ; \quad W_\mu^3 \tag{2.34}$$

Where as we have seen, $W_\mu^\pm$ have electromagnetic charge ± 1, and W^3 will be neutral, having interactions with fermions

$$\overline{\psi}_i \gamma^\mu W_\mu^3 \sigma_{ij}^3 \psi_j \tag{2.35}$$

and σ^3 is diagonal. A priori, W_μ^3 is then a candidate photon. This is not possible. In the first place, only left-handed fermions interact with W_μ^3, therefore the right-handed states should be electromagnetically neutral. Also

$$\begin{pmatrix} \overline{\nu}_L & \overline{e}_L \end{pmatrix} \gamma^\mu W_\mu^3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} = (\overline{\nu}_L \gamma^\mu \nu_L - \overline{e}_L \gamma^\mu e_L) W_\mu^3 \tag{2.36}$$

And as we know, ν has charge zero.

We have to introduce another gauge boson that can, after symmetry breaking, give us a photon. That means another $U(1)$ group, and its generator should commute with the $SU(2)$ we postulated. This is the so-called hypercharge (Y) group, and its generator is

$$Y = 2(Q - T_{3L}) \tag{2.37}$$

where Q is the electromagnetic charge generator and T_{3L} is the diagonal $SU(2)$ generator. Our left-handed fermions are in the fundamental representation, therefore the eigenvalues of T_{3L} are $\pm 1/2$ for them , and 0 for the right-handed particles. With this choice, both components of each of the doublets we defined have the same hypercharge. We get

Field	Y/2
e_R	-1
u_R	$2/3$
d_R	$-1/3$
ℓ_R	$-1/2$
q_R	$1/6$
W_μ	0

We shall then build the SM as a theory based in the group $SU(3)_c \times SU(2)_L \times U(1)_Y$ which is spontaneously broken to $SU(3)_c \times U(1)_{em}$.

Chapter 3

Spontaneous symmetry breaking

3.1 Discrete symmetries

Take a scalar field ϕ with a Lagrangean

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi^2) \tag{3.1}$$

It is clearly invariant under a Z_2 symmetry:

$$\phi \to -\phi \tag{3.2}$$

Suppose that the potential has two global minima at $\phi = \pm v$, such as in

$$V(\phi) = \frac{\lambda}{4}(\phi^2 - v^2)^2 = -\frac{m_\phi^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 + const. \tag{3.3}$$

which represents a field with a purely imaginary mass

$$-m_\phi^2 \equiv -\lambda v^2 \tag{3.4}$$

The *vacuum expectation value* (vev) of this field will be the lowest energy configuration

$$\langle 0|\phi|0\rangle \equiv \langle\phi\rangle = \pm v \tag{3.5}$$

Contrary to the situation with a usual, real mass field, the vacuum expectation value for this field is not zero. We can define a new field

$$\eta \equiv \phi - v \tag{3.6}$$

such that

$$\langle \eta \rangle = 0 \tag{3.7}$$

The Lagrangean for η is

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\partial_\mu \eta \partial^\mu \eta - \frac{\lambda}{4}(\eta^2 + 2\eta v)^2 \\ &= \frac{1}{2}\partial_\mu \eta \partial^\mu \eta - \frac{\lambda}{4}\left(\eta^4 + 4\eta^2 v^2 + 4\eta^3 v\right) \end{aligned} \tag{3.8}$$

In other words, we have a field with a mass $m_\eta^2 = 2\lambda v^2$, subject to a potential that no longer has a Z_2 symmetry, it is broken by the cubic terms. However, notice that we have not arrived to the most general potential for a real scalar field without any symmetry: there is instead a precise relation between the coefficients of the different terms. This is what makes it possible to define a field ϕ (the original one) with a Z_2-symmetric potential. However, when we do that we loose the condition that the field has a zero vev, and we cannot quantize the theory around such a vacuum. This phenomenon is called *spontaneous symmetry breaking*, or SSB. The symmetry is there, only it manifests itself in a different way than usual.

SSB of a discrete symmetry has very interesting consequence (such as the possibility of producing domain walls), however we are interested in spontaneous breakdown of continuos symmetries, so let us move on.

3.2 Abelian case: $U(1)$

Let us now take a complex scalar field

$$\phi = \phi_1 + i\phi_2 \tag{3.9}$$

with a Lagrangean

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^* \partial^\mu \phi - V(\phi^* \phi) \tag{3.10}$$

It is invariant under phase transformations –the $U(1)$ group

$$\phi \to e^{i\theta}\phi \tag{3.11}$$

Again, we can write a potential with global minima outside the origin, such as

$$V(\phi^*\phi) = \frac{\lambda}{4}(\phi^*\phi - v^2)^2 \tag{3.12}$$

The set of minima of this potential forms a circle:

$$\langle\phi^*\phi\rangle = v^2 \Rightarrow \langle\phi\rangle = e^{i\alpha}v \tag{3.13}$$

with $\alpha = 0...2\pi$ arbitrary. Without loosing generality, suppose now $\alpha = 0$ (all vacua parametrized by α are equivalent), that is to say, take a real vev for ϕ:

$$\langle\phi_1\rangle = v \; ; \quad \langle\phi_2\rangle = 0 \tag{3.14}$$

As in the discrete symmetry case, we can define a zero-vev shifted field η

$$\eta_1 = \phi_1 - v \; ; \quad \eta_2 = \phi_2 \tag{3.15}$$

In terms of this new field we get

$$V = \frac{\lambda}{4}(\eta_1^2 + 2\eta_1 v + \eta_2^2)^2 \tag{3.16}$$

or

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\left(\partial_\mu\eta_1\partial^\mu\eta_1 + \partial_\mu\eta_2\partial^\mu\eta_2\right) \\ &- \frac{\lambda}{4}\left(\eta_1^4 + \eta_2^4 + 4\eta_1^2 v^2 + 2\eta_1^2\eta_2^2 + 4\eta_1^3 v\right) \end{aligned} \tag{3.17}$$

We no longer see the $U(1)$ symmetry, but the imaginary mass fields have disappeared. We have now two fields with masses

$$m(\eta_1) = 2\lambda v^2; \quad m(\eta_2) = 0 \tag{3.18}$$

η_2, the massless field, *Nambu-Goldstone boson*.

3.3 Non-abelian case: $SO(n)$

The next step is to consider a non-abelian group. Let us take $SO(n)$, rotations in a n-dimensional space, as an example. In the previous section we have in fact seen a particular case, $U(1) \sim SO(2)$, so it will be an easy task to extend the discussion. Consider a scalar field with n real components, transforming under the fundamental representation of $SO(n)$

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{pmatrix} \tag{3.19}$$

And a Lagrangean

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^T \partial^\mu \phi - V(\phi^T \phi) \qquad V = \frac{\lambda}{4}(\phi^T \phi - v^2)^2 \tag{3.20}$$

so that

$$\langle \phi^T \phi \rangle = v^2 \tag{3.21}$$

There is a manifold of equivalent vacua, connected by a $SO(n)$ transformation. Choosing one of them

$$\langle \phi^a \rangle = \delta_0^a \tag{3.22}$$

(with $a = 0...n$), we can define the components of the shifted field η

$$\eta_0 = \phi_0 - v \qquad \eta_b = \phi_b \quad (b \neq 0) \tag{3.23}$$

getting

$$\mathcal{L} = \frac{1}{2}\partial_\mu \eta^a \partial^\mu \eta_a - \frac{\lambda}{4}(\eta_0^2 + 2\eta_0 v + \eta_1^2 + \eta_2^2 + ...)^2 \tag{3.24}$$

Notice that the η_b fields, con $b = 1..n$ have

$$\mathcal{L} = \frac{1}{2}\partial_\mu \eta^b \partial^\mu \eta_b - \frac{\lambda}{4}(\eta^b \eta_b)^2 \tag{3.25}$$

that is, a $SO(n-1)$ symmetry survives. We have in this case $n-1$ massless Nambu-Goldstone bosons.

Goldstone theorem predicts exactly how many Nambu-Goldstone bosons we get when a global symmetry is broken spontaneously. Before giving it, let us see an illustrative example, $SO(3)$.

$SO(3)$

The generators in the fundamental representation of $SO(3)$ are

$$(J_i)_{jk} = -i\epsilon_{ijk} \tag{3.26}$$

$$J_1 = -i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad J_2 = -i\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad J_3 = -i\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.27}$$

And choosing the vev in the direction

$$\langle\phi\rangle = \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix} \tag{3.28}$$

we get

$$J_1[\langle\phi\rangle] = 0 \quad J_2[\langle\phi\rangle] \neq 0 \quad J_3[\langle\phi\rangle] \neq 0 \tag{3.29}$$

Goldstone Theorem *There is one Nambu-Goldstone boson for each generator that does not annihilate the vacuum*

Only one generator, J_1, annihilates the vacuum. That is to say, the $U(1)$ group elements generated by it will leave the vacuum invariant

$$U\langle\phi\rangle \equiv e^{i\theta J_1}\langle\phi\rangle = \langle\phi\rangle \tag{3.30}$$

The vacuum is still invariant under some of the group elements, those corresponding to the subgroup that "survives" SSB. In this case

$$SO(3) \to SO(2) \sim U(1) \tag{3.31}$$

whose only generator we identify with J_1.

It is straightforward to extend this to a $SO(n)$ group broken by the vev of a field in the fundamental representation. We will have

$$SO(n) \to SO(n-1) \tag{3.32}$$

Out of the $n(n-1)/2$ generators of $SO(n)$, we say that $(n-1)(n-2)/2$ survive (those of $SO(n-1)$), in the sense that they still annihilate the vacuum, and we will get

$$\frac{n(n-1)}{2} - \frac{(n-1)(n-2)}{2} = n-1$$

Nambu-Goldstone bosons.

This results are easily generalized to other compact groups such as the one we need to build the SM. Let us see the $SU(2)$ case

$SU(2)$

Take a scalar field, $SU(2)$ doublet

$$\Phi = \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix} \tag{3.33}$$

and a SSB potential

$$V(\Phi^\dagger\Phi) = \frac{\lambda}{4}(\Phi^\dagger\Phi - v^2)^2 \tag{3.34}$$

We choose the vev in the direction

$$\langle\Phi\rangle = \begin{pmatrix} v \\ 0 \end{pmatrix} \tag{3.35}$$

so that

$$\eta = \begin{pmatrix} \eta_1 - v + i\eta_2 \\ \eta_3 + i\eta_4 \end{pmatrix} \tag{3.36}$$

and we will get three massless fields, η_2, η_3, η_4. On the other hand, the generators of $SU(2)$ are the Pauli matrices, and it can be easily seen that none of them annihilates the vacuum: all three generators are "broken", and again we see the Goldstone theorem in action. We have spontaneously broken $SU(2)$ with a fundamental representation, and we have broken it completely:

$$SU(2) \to 0 \tag{3.37}$$

3.4 Degeneracy

When a symmetry group is spontaneously broken, the degeneracy of the energy eigenstates that live tin the group representation is lifted. Let U be an element of the group G, commuting with the Hamiltonian

$$UH_0U^\dagger = H_0 \tag{3.38}$$

And take two Hamiltonian eigenstates connected by a group transformation

$$|A\rangle\ ; \quad |B\rangle = U|A\rangle \tag{3.39}$$

It follows that

$$E_A = \langle A|H_0|A\rangle = \langle B|H_0|B\rangle = E_B \tag{3.40}$$

As expected, two eigenstates transforming under the same group representation have the same energy (as do for example the components of an $SU(2)$ doublet). Now, these states can be defined in terms of creation operators acting on the vacuum:

$$|A\rangle = \hat{O}_A|0\rangle \;; \quad |B\rangle = \hat{O}_B|0\rangle \tag{3.41}$$

and we will have

$$\hat{O}_B = U\hat{O}_A U^\dagger \tag{3.42}$$

Then, from 3.39

$$\hat{O}_B|0\rangle = U\hat{O}_A|0\rangle = U\hat{O}_A U^\dagger U|0\rangle = \hat{O}_B U|0\rangle \tag{3.43}$$

That is to say, we need the element U to leave invariant the vacuum

$$|0\rangle = U|0\rangle \tag{3.44}$$

If this does not happen, the degeneracy $E_A = E_B$ will be lifted. In the case of a completely broken $SU(2)$, the components of the doublet will have different masses, precisely what we wanted for the SM.

Box 3.1: Goldstone Theorem

Let G be the symmetry group of a certain Lagrangean. Emmy Noether's theorem predicts the existence of a current and a conserved charge

$$\partial_\mu J^\mu = 0 \qquad Q(t) = \int d^3x J_0(\vec{x}, t) \tag{3.45}$$

Now consider any operator, function of the fields in the theory, $A(x)$. We have

$$\begin{aligned} 0 &= \int d^3x [\partial^\mu J_\mu(x,t), A(0)] \\ &= \partial^0 \int d^3x [J_0(x,t), A(0)] + \int d\vec{S}.[\vec{J}(x,t), A(0)] \end{aligned} \tag{3.46}$$

and we can make the last term to vanish by taking a large enough surface. Then

$$\frac{d}{dt}[Q(t), A(0)] = 0 \tag{3.47}$$

If the fields in the combination A are such that

$$\langle 0|\phi|0\rangle \neq 0 \tag{3.48}$$

we will also have

$$\langle 0|[Q(t), A(0)]|0\rangle \neq 0 \equiv \alpha \tag{3.49}$$

with α time-independent. Therefore

$$\langle 0| \int d^3x [J_0(x,t), A(0)]|0\rangle = \alpha \tag{3.50}$$

is also time-independent.

We now insert in this expression complete basis of Hamiltonian eigenstates, the translation operator, and integrate. We get

$$\begin{aligned} \alpha &= \sum_n (2\phi)^3 \delta^3(\vec{p}) \left\{ \langle 0|J_0(0)|n\rangle\langle n|A(0)|0\rangle e^{iE_n t} \right. \\ &\quad \left. - e^{-iE_n t}\langle 0|A(0)|n\rangle\langle n|J_0(0)|0\rangle \right\} \end{aligned} \tag{3.51}$$

This expression cannot be time-independent if

$$\langle 0|A(0)|n\rangle\langle n|J_0(0)|0\rangle \neq 0 \tag{3.52}$$

unless the state $|n\rangle$ has $E_n = 0$. In other words, if the charge Q does not annihilate the vacuum, there must exist a massless state. In the case of a non-abelian group, the reasoning can be repeated for each Q^a, so that for each generator that does not annihilate vacuum, there is a massless state, the Nambu-Goldstone boson .

Exercise 5
Use the adjoint representation to break $SU(2)$ spontaneously, find the Nambu-Goldstone bosons and state which symmetry survives.

Exercise 6
Consider a theory with an $SO(3)$ global symmetry, and two fields in the fundamental representation, ϕ_1 y ϕ_2. Write down a potential for these fields that forces both of them to take a vev, in such directions that $SO(3)$ gets completely broken.

Exercise 7
Break the $SU(5)$ group with an adjoint representation, omitting the cubic terms in the potential, and describe the different symmetry breaking patterns which are possible.

Chapter 4

Higgs mechanism

The spontaneous breakdown of a gauge symmetry is called the *Higgs mechanism*. Let us study the abelian and non-abelian cases separately.

4.1 Abelian case

Suppose we have theory with a scalar field transforming under gauged $U(1)$:

$$\phi \to e^{i\theta(x)}\phi \tag{4.1}$$

Its Lagrangean will be similar to 3.10, where the usual derivative is replaced by the covariant derivative

$$\partial_\mu\phi \to D_\mu\phi = (\partial_\mu - igA_\mu)\phi \tag{4.2}$$

where we have supposed that ϕ has unit charge under this symmetry. Furthermore, we have to add the kinetic terms for the gauge field.

Kinetic terms for $\phi = \phi_1 + i\phi_2$ will give

$$\begin{aligned}(D_\mu\phi)^*(D^\mu\phi) &= \partial_\mu\phi_1\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2 \\ &+ 2gA^\mu(\phi_1\partial_\mu\phi_2 - \phi_2\partial_\mu\phi_1) + g^2A_\mu A^\mu(\phi_1^2 + \phi_2^2)\end{aligned} \tag{4.3}$$

If we choose a symmetry breaking potential as in 3.12, we will have $\langle\phi\rangle = ve^{i\alpha}$, and again we can choose the phase to be zero. In terms of the shifted fields

$$\eta_1 = \phi_1 - v \qquad \eta_2 = \phi_2 \tag{4.4}$$

the kinetic terms 4.3 become

$$\begin{aligned}(D_\mu\eta)^*(D^\mu\eta) &= \partial_\mu\eta_1\partial^\mu\eta_1 + \partial_\mu\eta_2\partial^\mu\eta_2 \\ &+ 2gA^\mu(\eta_1\partial_\mu\eta_2 - \eta_2\partial_\mu\eta_1) + g^2A_\mu A^\mu(\eta_1^2+\eta_2^2) \\ &+ 2gvA^\mu\partial_\mu\eta_2 + 2g^2v\eta_1 + g^2v^2A_\mu A^\mu \end{aligned} \tag{4.5}$$

Notice that the last term is a *mass term for the gauge boson*:

$$M_A = gv \tag{4.6}$$

proportional to the ϕ vev. We have also that the field η_1 has a mass, just as in the global case,

$$m_\eta = 2\lambda v^2 \tag{4.7}$$

This is the so called *Higgs field*.

It seems that we have achieved our goal of finding a theory with a gauge symmetry with massive gauge bosons. However, we could think that this is not a physical effect, that is, that we can make the mass vanish with a gauge transformation.

Using gauge freedom, we can go to the so-called *unitary gauge*. If we write ϕ in polar form

$$\phi(x) = (\eta(x)+v)e^{iG(x)/v} \tag{4.8}$$

(where η is now a real field), we see that the $G(x)$ can be gauged away with a transformation $U = e^{-iG(x)/v}$, giving

$$\phi(x) = \eta(x) + v \tag{4.9}$$

And the full Lagrangean in this unitary gauge is

$$\begin{aligned}\mathcal{L}_U &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \partial_\mu\eta\partial^\mu\eta + \frac{m_A^2}{A}_\mu A^\mu + \frac{g^2}{2}A_\mu A^\mu(\eta^2+2\eta v) \\ &\quad -\frac{\lambda}{4}(\eta^4+4v\eta^3) - \frac{m_\eta^2}{2}\eta^2 \end{aligned} \tag{4.10}$$

We have a theory with

- a real scalar field with mass m_η
- a gauge field with mass m_A

The other real field, corresponding to the Nambu-Goldstone boson of a theory with a global symmetry, has disappeared: this is the Higgs mechanism. If we count the degrees of freedom in the theory written with ϕ or with η:

$\langle\phi\rangle \neq 0$		$\langle\eta\rangle = 0$	
$\phi_1 + i\phi_2$	$= 2$	η	$= 1$
$A_\mu\ (m_A = 0)$	$= 2$	$A_\mu\ (m_A \neq 0)$	$= 3$

Both cases have in total 4 degrees of freedom. The extra degree of freedom of the massive gauge boson comes from the real scalar field that has disappeared One says that the gauge boson has eaten the scalar G and grown heavy. G is often called a "would-be-Goldstone boson".

Now, as we have argued, the inclusion of mass terms for the gauge bosons breaks the gauge invariance of the theory, and it loses renormalizability. Is renormalizability lost here?

The answer is no. We have not arbitrarily broken the gauge symmetry by adding a mass term by hand, we have broken it spontaneously, and the mass of A_μ has a particular dependence on the parameters of the ϕ potential. The symmetry is not broken, it is spontaneously broken, or hidden, and this guarantees renormalizability.

4.2 Gauge invariance

Let us write ϕ as

$$\phi = \eta + v + iG \tag{4.11}$$

(The imaginary part of ϕ approximately coincides with G, the would-be Goldstone boson, if we expand $G \ll v$).

We add a general gauge term to the Lagrangean

$$\Delta\mathcal{L}_{calibre} = -\frac{1}{2}(\sqrt{\xi}\, g\, v\, G + \frac{1}{\sqrt{\xi}}\, \partial_\mu A^\mu)^2 \tag{4.12}$$

(which is equivalent to the relativistic gauge $v = 0$). Integrating by parts the total action

$$\begin{aligned}\mathcal{L} = \mathcal{L}(\eta) + \frac{1}{2}A_\mu \left[(\Box + m_A^2)g^{\mu\nu} - \left(1 - \frac{1}{\xi}\right)\partial^\mu\partial^\nu\right] A_\nu \\ -\frac{1}{2}G\left[\Box + \xi\, m_A^2\right] G\end{aligned} \tag{4.13}$$

It is evident here that the mass of G is gauge dependent, it disappears in Landau gauge when $\xi = 0$, so it is unphysical. A_μ, instead, has a physical mass. We calculate the propagators for A_μ and G

$$\Delta^\xi_{\mu\nu}(A) = \frac{-i}{k^2 - m_A^2}\left[g_{\mu\nu} + (\xi - 1)\frac{k_\mu k_\nu}{k^2 - m_A^2 \xi}\right] \tag{4.14}$$

$$D^\xi G = \frac{i}{k^2 - \xi m_A^2} \tag{4.15}$$

Now we can choose different gauges:

- Unitary gauge: $\xi \to \infty$

$$\Delta^\xi_{\mu\nu}(A) \to \frac{-i}{k^2 - m_A^2}\left[g_{\mu\nu} + \frac{k_\mu k_\nu}{m_A^2}\right] \tag{4.16}$$

$$D^\xi G \to 0 \tag{4.17}$$

The would-be Goldstone boson, as we already know, disappears in this gauge. The gauge field propagator, instead, has serious divergences for $k \to \infty$, as can be seen form the second term. In this gauge, the theory seems non-renormalizable.

- 'tHooft-Feynman gauge: $\xi = 1$

$$\Delta^\xi_{\mu\nu}(A) = \frac{-i g_{\mu\nu}}{k^2 - m_A^2} \tag{4.18}$$

$$D^\xi G = \frac{i}{k^2 - m_A^2} \tag{4.19}$$

We have a theory with a boson G with mass m_A, and the gauge boson propagator behaves for $k \to \infty$ as

$$\Delta^\xi_{\mu\nu}(A) \to \frac{-i g_{\mu\nu}}{k^2} \tag{4.20}$$

and will not give unremovable divergencies.

- Landau gauge: $\xi = 0$

$$\Delta^{\xi}_{\mu\nu}(A) = \frac{-i}{k^2 - m_A^2}\left[g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right] \tag{4.21}$$

$$D^{\xi}G = \frac{i}{k^2} \tag{4.22}$$

We have a massless boson G, and a gauge field propagator which is well-behaved for large momenta.

The last two gauges are often called *renormalizable* gauges, and they have an unphysical field G. If we wish to calculate processes, it is convenient to use these two, while the unitary is the most appropriate if we want to know about the physical fields and their masses. Fortunately, it is still a gauge theory, even if the symmetry is hidden (spontaneously broken). It can be proven rigorously that the scattering matrices of physical processes do not depend on ξ.

4.3 Non-abelian case

Let us again take the example of $SO(3)$, and ϕ in the fundamental representation: $\phi = (phi_1, \phi_2, \phi_3)$. We will have three gauge fields that transform under the adjoint representation, $A_\mu = (A^1_\mu, A^2_\mu, A^3_\mu)$. The covariant derivative will look like

$$D_\mu \phi_i = \partial_\mu \phi_i - igA^a_\mu (J^a)_{ij}\phi_j \tag{4.23}$$

where the J^a are those of 3.27. Suppose ϕ takes a vev in the direction 1 and define

$$\eta = \phi - \langle\phi\rangle = \phi - \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix} \tag{4.24}$$

We will have for each generator

$$J_1\phi = J_1\eta; \quad J_2\phi = J_2\eta + \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} J_3\phi = J_3\eta - \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix} \tag{4.25}$$

That is

$$D_\mu\phi = \partial_\mu\eta - igA^2_\mu \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} + igA^3_\mu \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix} \tag{4.26}$$

So that the kinetic term will contain

$$
\begin{aligned}
(D^\mu\phi)^\dagger(D_\mu\phi) = (D^\mu\eta)^\dagger(D_\mu\eta) + ... + \\
g^2v^2(A^2_\mu A^{\mu 2}) + g^2v^2(A^3_\mu A^{\mu 3})
\end{aligned}
\tag{4.27}
$$

Two of the gauge bosons get a mass –they correspond to the two Nambu-Goldstone bosons of the global theory. In this way, the Higgs mechanism, through the Goldstone theorem, guarantees the existence of *one massive gauge boson for each generator that does not annihilate vacuum* Only the gauge bosons of the group that survives symmetry breaking will remain massless, in this case it is the photon of $SO(2) \sim U(1)$. In the unitary gauge, the fields η_2 and η_3 disappear, and η_1 is called the Higgs.

One could ask what would have happened if $\langle\phi\rangle$ had been chosen in other direction, say 3. In this case, A^3_μ would have remained massless, and clearly this is just a meaningless change of name. In fact, if we had chosen

$$
\langle\phi\rangle = v \begin{pmatrix} \cos(\theta)\sin(\varphi) \\ \cos(\theta)\cos(\varphi) \\ \sin(\theta) \end{pmatrix}
\tag{4.28}
$$

we would still get a massless linear combination of the three A_μ, and two massive ones. In this case, a linear combination of the three generators J would survive (annihilate the vacuum), becoming the generator of the surviving symmetry.

We have seen that the fundamental representation of gauged $SO(3)$ can break the symmetry down to gauged $U(1)$: electromagnetism.

Exercise 8
Show that (4.14) and (4.15) are indeed the propagators for the gauge field and the N-G boson in this theory

Chapter 5

Higgs mechanism: adding the fermions

Because the vev of the Higgs field takes non vanishing values, it must be a Lorentz scalar (otherwise it would break this symmetry also). This means that we have to add a spin 0 boson to the particle spectrum of the SM, with a Lagrangean compatible with the internal symmetries that we have postulated. We must also make sure that the component of the field that takes a vev transforms trivially with respect to the gauge groups that we want preserved in the final theory: $SU(3)_c$ and $U(1)_{em}$. The most general Lagrangean in a theory with several fermionic fields ψ_i and a Higgs boson should include the so-called *Yukawa interaction terms*, of the form

$$\mathcal{L}_Y = y_{ij}\overline{\psi}_i\,\psi_j\phi \tag{5.1}$$

where y_{ij} is matrix of Yukawa coupling constants. If we have SSB, in terms of the shifted field

$$\eta = \phi - v \tag{5.2}$$

$\mathcal{L}_Y$ will induce a mass for the fermions:

$$y_{ij}\,v\,\overline{\psi}_i\psi_j \tag{5.3}$$

Let us see an abelian example.

5.1 Abelian case, local symmetry

Suppose we have a $U(1)$ symmetry that does not respect parity, under which fermions transform as

$$\psi_L \to e^{iq/2}\psi_L \; ; \quad \psi_R \to e^{-iq/2}\psi_R \tag{5.4}$$

This symmetry forbids a Dirac mass term $\overline{\psi}_L \psi_R$ [1]. Suppose that the Higgs field has twice the charge as the fermions

$$\phi \to e^{iq}\phi \tag{5.5}$$

and let us write the most general SSB Lagrangean with this symmetry

$$\begin{aligned} \mathcal{L} &= i\overline{\psi}_L \gamma^\mu \partial_\mu \psi_L + \overline{\psi}_R \gamma^\mu \partial_\mu \psi_R + \frac{1}{2}\partial_\mu \phi^* \partial^\mu \phi \\ &- y(\overline{\psi}_L \, \phi \, \psi_R + \overline{\psi}_R \, \phi^* \, \psi_L) - \frac{\lambda}{4}(\phi^* \phi - v^2)^2 \end{aligned} \tag{5.6}$$

where we have restricted to the one fermion case. Setting

$$\phi = \eta + v + iG \tag{5.7}$$

the Yukawa term gives

$$\begin{aligned} & \overline{\psi}_L \, \phi \, \psi_R + \overline{\psi}_R \, \phi^* \, \psi_L \\ = \; & (\overline{\psi}_L \psi_R + \overline{\psi}_R \psi_L)(\eta + v) + i(\overline{\psi}_L \psi_R - \overline{\psi}_R \psi_L)G \end{aligned} \tag{5.8}$$

and rewriting $\mathcal{L}$ we get

$$\begin{aligned} \mathcal{L} &= i\overline{\psi}\gamma^\mu \partial_\mu \psi + \frac{1}{2}\partial_\mu \eta \partial^\mu \eta + \frac{1}{2}\partial_\mu G \partial^\mu G \\ &- m_\psi \overline{\psi}\psi - \frac{1}{2}m_\eta^2 \eta^2 - \frac{m_\psi}{v}\overline{\psi}\psi\eta - i\frac{m_\psi}{v}\overline{\psi}\gamma_5 \psi G \\ &- \frac{\lambda}{4}(\eta^2 + G^2)(\eta^2 + G^2 + 2\eta v) \end{aligned} \tag{5.9}$$

where we have defined

$$m_\psi \equiv yv \; ; \quad m_\eta^2 = 2\lambda v^2 \tag{5.10}$$

[1] As we have argued, any internal symmetry based on a compact Lie group forbids a Majorana mass term.

We have thus changed a theory with massless fermions into one with massive fermions having a pseudoscalar coupling with a Nambu-Goldstone boson. This can be a serious problem: the coupling with a massless particle leads to a interaction between fermions with infinite range, that could eventually compete with other infinite-range forces such as gravitation. Even if this interaction is very weak, it can become important in systems with a large number of fermions, such as in astrophysical objects.

Box 5.1: Pseudoscalar interactions

A star could in principle disintegrate completely by emitting Nambu-Goldstone bosons. We will see that this does not happen. Consider non-relativistic fermions with

$$m_\phi \gg |\vec{p}_\phi| \tag{5.11}$$

In the Dirac basis, this fermions can be written as

$$\psi = \begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{2m} \end{pmatrix} u = \begin{pmatrix} 1 \\ \ll 1 \end{pmatrix} u \tag{5.12}$$

and in this basis

$$\gamma_5 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{5.13}$$

So that a coupling with G will give a diagram as in figure 5.1. We have

$$\frac{m_\psi}{v} G\overline{\psi}\gamma_5\psi = \frac{m_\psi}{v}\, G\, u^\dagger(\vec{p}')\left(\frac{\vec{\sigma}\cdot(\vec{p}'-\vec{p})}{2m_\psi}\right)u(\vec{p}) \tag{5.14}$$

The interaction is spin dependent*. If we take a large number of fermions, coupling with G will be proportional to the mean value of the spin*

$$\frac{1}{v}G\langle\vec{s}\rangle\cdot\vec{q} \tag{5.15}$$

For a large system as a star, $\sim 10^{50}$ *baryons, spins cancel out and* $\langle\vec{s}\rangle \sim 0$*. The emission of N-G bosons is negligible in comparison*

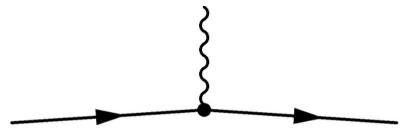


Figure 5.1: Fermion-pseudoscalar interaction.

with the gravitational interaction of $\sim 10^{50}$ *baryons. Nevertheless, if there is a Nambu-Goldstone boson in nature (such as the axion), stars would emit them, and this can be used to set limits to its mass.*

Notice that charge assignment is important. Had we set

$$Q(\psi_L) = Q(\psi_R) = \frac{1}{2}\,; \quad Q(\phi) = -1 \tag{5.16}$$

we would have obtained a direct mass term for the fermions, and no Yukawa coupling. This way, symmetries can become a mechanism for controlling fermion masses to fit our needs.

5.2 Abelian case, $U(1)$ local

The $U(1)$ local case does not add major difficulties. In the unitary gauge, we can make G disappear, so that $\phi = \eta + v$, and we still get a mass for the fermions. However, we will take a look at the gauge invariance of the results.

Box 5.2: Gauge invariance- fermion scattering

Once again we add

$$\Delta\mathcal{L}_{calibre} = -\frac{1}{2}(\sqrt{\xi}gvG + \frac{1}{\sqrt{\xi}}\partial_\mu A^\mu)^2 \tag{5.17}$$

so that the fermionic part of $\mathcal{L}$ *is now*

$$\begin{aligned}\mathcal{L}_\psi &= i\overline{\psi}_L\gamma^\mu D_\mu\psi_L + i\overline{\psi}_R\gamma^\mu D_\mu\psi_R + y\overline{\psi}_L\phi\psi_R + h.c.\\ &= i\overline{\psi}_L\gamma^\mu(\partial_\mu - i\frac{g}{2}A_\mu)\psi_L + i\overline{\psi}_R\gamma^\mu(\partial_\mu + i\frac{g}{2}A_\mu)\psi_R\\ &\quad + y\overline{\psi}_L\phi\psi_R + h.c.\\ &= i\overline{\psi}\gamma^\mu\partial_\mu\psi + \frac{g}{2}\overline{\psi}\gamma^\mu\gamma_5\psi A_\mu + y\overline{\psi}_L\phi\psi_R + h.c. \end{aligned} \tag{5.18}$$

that is, with $\phi = \eta + v + iG$,

$$\begin{aligned}\mathcal{L}_\psi &= m_\psi\overline{\psi}\psi + i\overline{\psi}\gamma^\mu\partial_\mu\psi - g\frac{m_\psi}{m_A}\eta\overline{\psi}\psi\\ &\quad -ig\frac{m_\psi}{m_A}G\overline{\psi}\gamma_5\psi - \frac{g}{2}\overline{\psi}\gamma^\mu\gamma_5\psi A_\mu \end{aligned} \tag{5.19}$$

where

$$m_A = gv\ ; \quad m_\psi = yv \tag{5.20}$$

The diagrams depending on ξ *in fermion scattering come from terms in the last line of 5.19*

With the propagators 4.14, 4.15, we calculate the amplitudes

$$\begin{aligned}\mathcal{M}(a) &= \left(\frac{ig}{2}\right)^2(\overline{\psi}_1\gamma^\mu\gamma_5\psi_2)(\overline{\psi}_3\gamma^\nu\gamma_5\psi_4)\frac{-i}{k^2 - m_A^2}\left(g_{\mu\nu} + (\xi - 1)\frac{k_\mu k_\nu}{k^2 - \xi m_A^2}\right)\\ &= \frac{ig^2}{4(k^2 - m_A^2)}(\overline{\psi}\gamma^\mu\gamma_5\psi)^2 - \frac{(\xi - 1)(\overline{\psi}\gamma_5\psi)^2}{(k^2 - m_A^2)(k^2 - \xi m_A^2)}m_\psi^2 \end{aligned} \tag{5.21}$$

where we have used

$$\not{p}_i\psi = m_\psi\psi\ ; \quad k = p_1 - p_2 = p_4 - p_3 \tag{5.22}$$

$$\mathcal{M}(b) = \frac{ig^2 m_\psi^2}{m_A^2(k^2 - m_A^2\xi)}(\overline{\psi}\gamma_5\psi)^2 \tag{5.23}$$

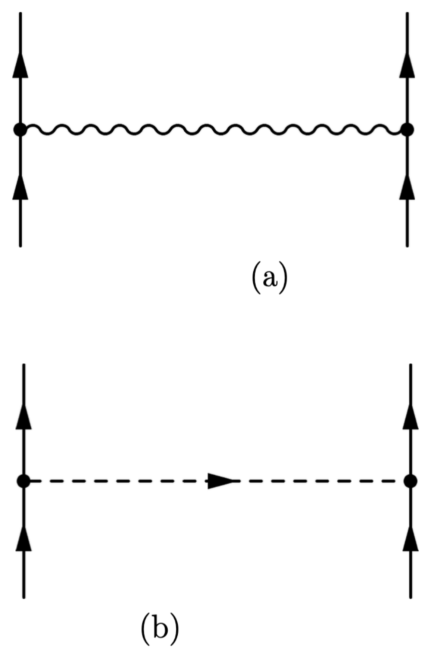


Figure 5.2: Fermion scattering by A_μ and G

So that the total amplitude

$$\mathcal{M} = \frac{ig^2}{(k^2 - m_A^2)} \left[\frac{1}{4} (\overline{\psi} \gamma_\mu \gamma_5 \psi)^2 + \frac{m_\psi^2}{m_A^2} (\overline{\psi} \gamma_5 \psi)^2 \right] \tag{5.24}$$

is ξ independent. *This result can be extended to all orders in perturbation theory.*

5.3 Non-abelian case, local symmetry

Let us take the example of a theory with $SU(2)$ symmetry, with fermions and Higgs field in the fundamental representation,

$$\Phi' = e^{i\theta_a(x)\sigma^a/2} \Phi \tag{5.25}$$

If the field has a potential like 3.34, its vev, which we choose

$$\begin{pmatrix} 0 \\ v \end{pmatrix} \tag{5.26}$$

completely breaks the symmetry as we have seen, $\sigma_a \langle \Phi \rangle \neq 0$.

Instead of 3.33, let us write Φ in terms of the field η and of three fields G_a

$$\Phi = e^{iG_a\sigma^a/2} \begin{pmatrix} 0 \\ \eta + v \end{pmatrix} \tag{5.27}$$

so that in unitary gauge

$$\Phi = \begin{pmatrix} 0 \\ \eta + v \end{pmatrix} \tag{5.28}$$

¿From the covariant derivatives we get

$$g^2 A_\mu^a \left[\frac{\sigma_a}{2} \langle \Phi \rangle \right] A^{\mu b} \left[\frac{\sigma_b}{2} \langle \Phi \rangle \right] \tag{5.29}$$

giving mass terms for the three gauge bosons,

$$\frac{g^2 v^2}{4} A_\mu^a A^{\mu a} \tag{5.30}$$

Notice that off-diagonal terms are not allowed.

Fermions

We shall assign transformation properties to the fermions in such a way as to prevent a mass term, but allowing a Yukawa term. This assignment, as we have seen, explicitly breaks parity. Take the two fermion case and, say,

$$\psi_{1R}\,,\psi_{2R} \quad \text{singlets} \tag{5.31}$$

$$\psi_L = \begin{pmatrix} \psi_{1L} \\ \psi_{2L} \end{pmatrix} \quad \text{doublets} \tag{5.32}$$

The allowed Yukawa terms will be

$$y_1(\overline{\psi}_L\,\Phi\,\psi_{1R} + h.c.) + y_2(\overline{\psi}_L\,\Phi\,\psi_{2R} + h.c.) \tag{5.33}$$

In the unitary gauge

$$y_1 \begin{pmatrix} \overline{\psi}_{1L} & \overline{\psi}_{2L} \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} \psi_{1R} = y_1\, v\, \overline{\psi}_{2L}\, \psi_{1R} \tag{5.34}$$

$$y_2 \begin{pmatrix} \overline{\psi}_{1L} & \overline{\psi}_{2L} \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} \psi_{2R} = y_2\, v\, \overline{\psi}_{2L}\, \psi_{2R} \tag{5.35}$$

It seems impossible to obtain a mass term for the fermion ψ_1. However, one can write $SU(2)$ invariants other than those in 5.33, using "charge conjugation" in $SU(2)$. Notice that

$$\Phi^T (i\sigma_2)\,\Phi \tag{5.36}$$

is $SU(2)$ invariant. Defining:

$$\tilde{\Phi} = i\sigma_2\,\Phi^* \tag{5.37}$$

we will have

$$\langle\tilde{\Phi}\rangle = \begin{pmatrix} v \\ 0 \end{pmatrix} \tag{5.38}$$

and we can write two more Yukawa terms

$$y_3(\overline{\psi}_L\,\tilde{\Phi}\,\psi_{1R} + h.c.) + y_4(\overline{\psi}_L\,\tilde{\Phi}\,\psi_{2R} + h.c.) \tag{5.39}$$

giving all fermions in the theory a mass.

Box 5.3: Goldstone limit

Nambu-Goldstone bosons become, in the case of a local symmetry, in degrees of freedom for the gauge bosons that get a mass. Clearly, in a situation where one can neglect the gauge coupling,

$$g \to 0 \tag{5.40}$$

we should recover the massless N-G bosons. Taking into account 5.20, we can rewrite the amplitude for the fermion-fermion scattering

$$\mathcal{M} = \frac{ig^2}{(k^2 - g^2v^2)} \left[\frac{1}{4} (\overline{\psi}\gamma_\mu\gamma_5\psi)^2 + \frac{m_\psi^2}{g^2v^2} (\overline{\psi}\gamma_5\psi)^2 \right] \tag{5.41}$$

When $g \to 0$

$$\mathcal{M} = \frac{m_\psi^2}{v^2} (\overline{\psi}\gamma_5\psi)^2 \frac{1}{k^2}] \tag{5.42}$$

which would be the result of coupling $\overline{\psi}\gamma_5\psi$ *with a field with a propagator* $1/k^2$*: an axial coupling with a massless pseudovector. The Nambu-Goldstone boson (or bosons, the diagram must be repeated for each broken generator) has reappeared.*

Exercise 9
Explain why terms like $\Phi^\dagger\sigma_a\Phi\,\Phi^\dagger\sigma^a\Phi$ *are not included in the potential l (3.34) for a* $SU(2)$ *doublet.*

Exercise 10
Fill in the steps leading to (5.24).

Exercise 11
Take the Higgs doublet with a vev in the direction

$$\langle \Phi \rangle = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Show that the same pattern of boson and fermion masses is found.

Chapter 6

$SU(3)_c \times SU(2)_L \times U(1)_Y \rightarrow SU(3)_c \times U(1)_{em}$

We have now all the necessary ingredients to build the SM. We start including only the first family of fermions.

6.1 Fields

The symmetry group of the theory will be $SU(3)_C \times SU(2)_L \times U(1)_Y \times \Lambda$. The transformation properties of the fields in the theory can be summarized as

- Poincaré: chiral spinorial representations $\psi_{L(R)} = (1 \pm \gamma_5)/2\psi$, for fermions; vector bosons for the gauge fields. Invariants are those in the table 1.1.

- $SU(3)$: fundamental representation , only for quarks. The rest of the fields are singlets, except for the 8 gluons in the adjoint, G_μ. Invariants one can build

$$\overline{q}^\alpha q_\alpha \; ; \quad \overline{q}^\alpha \gamma_5 q_\alpha \; ; \quad \overline{q}^\alpha \gamma^\mu D_\mu q_\alpha \tag{6.1}$$

- $SU(2)_L$: fundamental representation, only for left-handed fermions ψ_L; right-handed fermions ψ_R are singlets; 3 gauge fields W_μ. Invariants one can build

$$i\psi_L \gamma^\mu D_\mu \psi_L \; ; \quad i\psi_R \gamma^\mu D_\mu \psi_R \tag{6.2}$$

Dirac mass terms are forbidden, but if we happen to have a neutral particle, a Majorana mass could be written.

- $U(1)_Y$: 1 neutral gauge boson B_μ, the rest of the particles will have charges assigned depending on its electromagnetic charge Q and its quantum number T_{3L} of $SU(2)$

$$Y = 2(Q - T_{3L}) \tag{6.3}$$

Invariants one can build: all neutral combinations.

Therefore, taking only the first family the theory will be built out of the following

	$SU(3)_C$	$SU(2)_L$	$Y/2$
e_R	S	S	-1
u_R	F	S	$2/3$
d_R	F	S	$-1/3$
ℓ_R	S	F	$-1/2$
q_R	F	F	$1/6$
G_μ	A	S	0
W_μ	S	A	0
B_μ	S	S	0

where S= singlet, F=fundamental, A=adjoint.

In addition, we have to choose a Higgs boson. Since we cannot write direct fermion mass terms, these must be vev-induced. As we have seen, in order to achieve this we need a doublet Higgs,

$$\Phi = \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix} \tag{6.4}$$

and it is easy to see that it has to have hypercharge

$$Y(\Phi) = 1 \tag{6.5}$$

so that we can write, e.g., $\overline{\ell}_L \Phi e_R$. Then, $\tilde{\Phi} = i\sigma_2 \Phi^*$ will have $Y(\tilde{\Phi}) = -1$.

Let us see how we can build invariants out of these fields.

6.2 Invariants

Kinetic terms

Covariant derivatives of fields in the fundamental representation will be

$$\begin{aligned} SU(3)_c &\rightarrow D_\mu = \partial_\mu - i\, c\, G^a_\mu \frac{\lambda_a}{2} \\ SU(2)_L &\rightarrow D_\mu = \partial_\mu - i\, g\, W^i_\mu \frac{\sigma_i}{2} \\ U(1)_Y &\rightarrow D_\mu = \partial_\mu - i\, g' \frac{Y}{2} B_\mu \end{aligned} \tag{6.6}$$

con $a = 1..8$ e $i = 1..3$. We therefore have, for the Higgs

$$D_\mu \Phi = (\partial_\mu - i\, g\, W_i \frac{\sigma^i}{2} - i\, g' \frac{1}{2} B_\mu)\Phi \tag{6.7}$$

For left-handed particles

$$\begin{aligned} D_\mu q_L &= (\partial_\mu - i\, c\, G^a_\mu \frac{\lambda_a}{2} - igW^i_\mu \frac{\sigma_i}{2} - i\, g' \frac{1}{6} B_\mu) q_L \\ D_\mu \ell_L &= (\partial_\mu - i\, g\, W^i_\mu \frac{\sigma_i}{2} + i\, g' \frac{1}{2} B_\mu) \ell_L \end{aligned} \tag{6.8}$$

For right-handed

$$\begin{aligned} D_\mu u_R &= (\partial_\mu - i\, c\, G^a_\mu \frac{\lambda_a}{2} - i\, g' \frac{4}{6} B_\mu) u_R \\ D_\mu d_R &= (\partial_\mu - i\, c\, G^a_\mu \frac{\lambda_a}{2} + i\, g' \frac{2}{6} B_\mu) d_R \\ D_\mu e_R &= (\partial_\mu - i\, g\, W^i_\mu \frac{\sigma_i}{2} + i\, g' B_\mu) e_R \end{aligned} \tag{6.9}$$

In the following we shall omit $SU(3)_C$ gauge terms. For each kind of gauge bosons, we give the corresponding kinetic terms

$$F_{\mu\nu^a} F_{\mu\nu_a}\, ; F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\mu A^a_\nu + g\, f_{abc} A^b_\mu A^c_\nu \tag{6.10}$$

which will include, in the non-abelian case, interactions among gauge bosons. We shall call them $G_{\mu\nu}$ (for $SU(3)_C$), $W_{\mu\nu}$ (for $SU(2)_L$) and $B_{\mu\nu}$ (for $U(1)_Y$).

Mass terms

Dirac mass terms

$$\overline{\psi}_L \psi_R \, ; \quad \overline{\psi}_R \psi_L \tag{6.11}$$

are forbidden by $SU(2)_L \times U(1)_Y$, while

$$\overline{\psi}_R \psi_R \tag{6.12}$$

are forbidden by Lorentz symmetry, and the possible Majorana mass terms

$$\psi_R^T C \psi_R \tag{6.13}$$

are forbidden by $U(1)_Y$. The only field allowed to have a mass is the Higgs, but

$$m_\Phi^2 \Phi^\dagger \Phi \tag{6.14}$$

will have the wrong sign in order to give SSB, and is not a physical mass. And gauge symmetry forbids masses for the gauge bosons. *There are no mass terms* in this Lagrangean.

Yukawa terms

Quarks and leptons are fundamentally distinguished by $SU(3)$. This fact is behind the conservation of two quantum numbers: B, baryon number, such that $B(q) = 1/3, B(\ell) = 0$, and L, lepton number, such that $L(q) = 0, L(\ell) = 1$. In other words, we will not have quark-lepton interaction terms in the Lagrangean.

For the quarks we have ($\alpha = 1...3$ of $SU(3)$):

$$y_d \, \overline{q}_L^\alpha \, \Phi \, d_{R\alpha} + h.c. \, ; \quad y_u \, \overline{q}_L^\alpha \, \tilde{\Phi} \, u_{R\alpha} + h.c. \tag{6.15}$$

and for leptons, just

$$y_e \, \overline{\ell}_L \, \Phi \, e_R + h.c. \tag{6.16}$$

Higgs Potential

The only other interactions allowed are Higgs self-interactions. We choose the most general potential with SSB

$$V(\Phi^\dagger \Phi) = \frac{\lambda}{4} (\Phi^\dagger \Phi - v^2)^2 \tag{6.17}$$

and call $m_\Phi^2 = \lambda v^2$

Standard Model Lagrangean

$$\begin{aligned}\mathcal{L}_{SM} &= -\frac{1}{4}(W_i^{\mu\nu}W^i_{\mu\nu} + G_a^{\mu\nu}G^a_{\mu\nu} + B^{\mu\nu}B_{\mu\nu} + \frac{1}{2}(D^\mu\,\Phi)^\dagger(D_\mu\,\Phi)\\ &+ i\bar{\ell}_L\gamma^\mu D_\mu\ell_L + i\bar{q}^\alpha_L\gamma^\mu D_\mu q_{L\alpha} + i\bar{u}^\alpha_R\gamma^\mu D_\mu u_{R\alpha} + i\bar{d}^\alpha_R\gamma^\mu D_\mu d_{R\alpha} + i\bar{e}_R\gamma^\mu D_\mu e_R\\ &+ (y_d\bar{q}^\alpha_L\,\Phi\,d_{R\alpha} + y_u\bar{u}^\alpha_L\,\tilde{\Phi}\,u_{R\alpha} + y_e\bar{\ell}_L\,\Phi\,e_R + h.c.) - V(\,\Phi^\dagger\Phi\,) \qquad (6.18)\end{aligned}$$

6.3 Symmetry breaking

According to the assignment 6.3, the $T_{3L} = -1/2$ components of the Higgs field 6.4

$$\phi_3 + i\phi_4 \qquad (6.19)$$

will have $Q = 0$. We can choose Φ in that direction without loss of generality[1], so that the $U(1)_{em}$ generator is preserved. With

$$\langle\Phi\rangle = \begin{pmatrix} 0 \\ v \end{pmatrix} \qquad (6.20)$$

we will have

$$\begin{aligned} T_{i\,L}\langle\Phi\rangle &= \frac{\sigma_i}{2}\langle\Phi\rangle \neq 0\\ \frac{Y}{2}\langle\Phi\rangle &= \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\langle\Phi\rangle \neq 0 \end{aligned} \qquad (6.21)$$

All generators are "broken". However, a linear combination of these generators

$$Q\langle\Phi\rangle = (T_{3L} + \frac{Y}{2})\langle\Phi\rangle = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}\langle\Phi\rangle = 0 \qquad (6.22)$$

annihilates the vacuum. Its the generator of the surviving symmetry, electromagnetism. We have succeeded in breaking

$$\begin{aligned} SU(2)_L \times U(1)_Y &\rightarrow U(1)_{em}\\ 3gen. + 1gen. &\rightarrow 1gen. \end{aligned} \qquad (6.23)$$

and we therefore have

[1] A different choice would just redefine Q

3 broken gen. = 3 massive gauge bosons

1 gen. that annihilates vacuum = 1 massless gauge boson: the photon

In order to identify these bosons, it is enough to look at the kinetic term of (the shifted) Φ and look for the gauge bosons that get a mass.

We define as usual

$$W^a_\mu T_a = W^+\sigma^+ + W^-\sigma^- + W^3\sigma^3 \tag{6.24}$$

and calculate

$$\begin{aligned} D_\mu\langle\Phi\rangle &= +\left(-igW^i_\mu\frac{\sigma_i}{2} - ig'\frac{1}{2}B_\mu\right)\langle\Phi\rangle \\ &= -\frac{i}{2}\begin{pmatrix} gW^3_\mu + g'B_\mu & g\sqrt{2}W^+_\mu \\ g\sqrt{2}W^-_\mu & -gW^3_\mu + g'B_\mu \end{pmatrix}\begin{pmatrix} 0 \\ v \end{pmatrix} \\ &= -\frac{i}{2}v\begin{pmatrix} g\sqrt{2}W^+_\mu \\ -gW^3_\mu + g'B_\mu \end{pmatrix} \end{aligned} \tag{6.25}$$

The combination $-gW^3_\mu + g'B_\mu$ is usually called Z_μ, and is written in terms of θ_W, the weak angle,

$$\tan\theta_W \equiv \frac{g'}{g} \tag{6.26}$$

as

$$\begin{aligned} Z_\mu &\equiv gW^3_\mu - g'B_\mu \\ &= \sqrt{g^2 + g'^2}(W^3_\mu \cos\theta_W - B_\mu \sin\theta_W) \end{aligned} \tag{6.27}$$

Then, the kinetic term will contain the mass terms

$$|D_\mu\langle\Phi\rangle|^2 = v^2\frac{g^2}{4}2W^{\mu+}W^-_\mu + v^2\frac{g^2}{4\cos^2\theta_W}Z^\mu Z_\mu. \tag{6.28}$$

We have then that the combination orthogonal to Z_μ

$$A_\mu \equiv \sqrt{g^2 + g'^2}(W^3_\mu \sin\theta_W + B_\mu \cos\theta_W) \tag{6.29}$$

will be massless, this will be our photon. Summarizing, we get as a result of SSB

	Q	m^2
A_μ	0	$m_A^2 = 0$
Z_μ	0	$m_Z^2 = v^2 g^2/4\cos^2\theta_W$
$W_\mu^\pm$	± 1	$m_W^2 = g^2 v^2/4$

6.4 Neutral currents

The $SU(2)_L \times U(1)_Y \to U(1)_{em}$ theory with one family of fermions hat we have constructed has the observed weak interactions with the charged bosons $W_\mu^\pm$, and electromagnetism. But we have a new boson, Z_μ^0. When the SM was proposed, neutral currents in weak interactions had not been observed, and their discovery was a spectacular achievement.

The coupling of fermions to neutral bosons is

$$\mathcal{L}_\psi^0 = \overline{\psi}\gamma^\mu\left(\frac{g}{2}\sigma_3 W_\mu^3 + \frac{g'}{2} Y B_\mu\right)\psi \tag{6.30}$$

Now using

$$W^3 = A\sin\theta_W + Z\cos\theta_W \ ; \quad B = A\cos\theta_W - Z\sin\theta_W \tag{6.31}$$

and defining

$$e \equiv g\sin\theta_W = g'\cos\theta_W \tag{6.32}$$

we get

$$\begin{aligned}\mathcal{L}_\psi^0 &= A_\mu\overline{\psi}\gamma^\mu\left(\frac{\sigma_3}{2} + \frac{Y}{2}\right)e\psi \\ &\quad + Z_\mu\overline{\psi}\gamma^\mu\left(\cos^2\theta_W\frac{\sigma_3}{2} - \sin^2\theta_W\frac{Y}{2}\right)\end{aligned} \tag{6.33}$$

The first term is just the electromagnetic interaction, since the generator in parenthesis is no other than Q. We have in addition

$$\mathcal{L}_Z = \frac{g}{\cos\theta_W} Z_\mu\overline{\psi}\gamma^\mu\left(T_{3L} - Q\sin^2\theta_W\right)\psi = \frac{g}{\cos\theta_W} Z_\mu J^{\mu 0} \tag{6.34}$$

an interaction felt by all charged fermions, even if they are $SU(2)_L$ singlets.

The first predictions of the model are then

- existence of neutral currents $J^{\mu 0}$
- existence of a neutral gauge boson Z_μ, with mass

$$m_Z = \frac{1}{\cos\theta_W} m_W$$

Detecting neutral currents allows measurements of θ_W in 4-fermion processes. The mass of Z was also determined, and the result is

$$\begin{aligned} m_W &= 80 GeV \\ m_Z &= 90 GeV \\ \sin^2\theta_W &= 0.23 \end{aligned} \tag{6.35}$$

which is a spectacular success of the theory.

6.5 Fermion masses

In the one family case, the fermion mass terms are very simple and come from the three Yukawa terms

$$y_e\, v\, \overline{e}_L e_R + y_d\, v\, \overline{d}_L d_R + y_u\, v\, \overline{u}_L u_R + h.c. \tag{6.36}$$

there is no neutrino mass. The inclusion of other families, however, greatly complicates this picture. But it will allow us to give an explanation of the only feature of weak interaction that we have listed and still remains unaccounted for: weak interactions mix quarks from different families, but not leptons.

Box 6.1: Higgs search

The same Yukawa term giving fermions a mass gives the interaction with the Higgs field, that we have called η*,*

$$\mathcal{L}_Y = g\frac{m_f}{\sqrt{2}M_W}\eta \overline{f}\, f \equiv h_f\, \eta\, \overline{f}\, f \tag{6.37}$$

Figure 6.1: Higgs production in e^+e^-

For the first family, with masses of order MeV, the coupling constant h_f is of order 10^{-4}, which makes it difficult to observe the Higgs in electron-positron (e^+e^-) collisions.

But the Higgs also interacts with the gauge bosons. For example the term

$$\frac{1}{2}\sqrt{g^2 + g'^2} M_Z Z_\mu Z^\mu \eta \tag{6.38}$$

contributes to the diagram e^+e^- of figure 6.1

Processes like this one, that can be observed for example in LEP, combined with data from high precision tests allow a limit

$$100GeV \leq m_H \leq 200GeV \tag{6.39}$$

Exercise 12
Take the Higgs doublet in the direction

$$\langle \Phi \rangle = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

con $v^2 = v_1^2 + v_2^2$. *Show that there is a massless photon, a neutral massive gauge boson (Z) and two charged ones ($W^\pm$). Find the eigenstates and its masses, showing that physical results are independent of the choice of vev.*

Exercise 13
Calculate the quartic and cubic gauge boson interactions. Show that there is no self-interaction term for the photon.

Chapter 7

Fermion families

As a consequence of SSB, we have the following fermionic currents

$$J_\mu^+ = \overline{\nu}_L \gamma_\mu e_L + \overline{u}_L \gamma_\mu d_L \tag{7.1}$$

$$J_\mu^0 = \sum_f \overline{f}\, \gamma_\mu (T_{3L} - Q \sin^2 \theta_W)\, f \tag{7.2}$$

$$J_\mu^{em} = \sum_f \overline{f}\, \gamma_\mu Q\, f \tag{7.3}$$

where until now $f = e_L, e_R, \nu_L, d_L, d_R, u_L, u_R$.

The mass terms allowed by symmetry are diagonal

$$y_i\, v\, \overline{f}_i\, f_i \tag{7.4}$$

but this will of course change when we introduce the other two families, formed by particles exactly replicating the quantum numbers of the first one

$$q_{Li} \; : \quad \begin{pmatrix} u_L \\ d_L \end{pmatrix} \; ; \quad \begin{pmatrix} c_L \\ s_L \end{pmatrix} \; ; \quad \begin{pmatrix} t_L \\ b_L \end{pmatrix} \tag{7.5}$$

$$\ell_{Li} \; : \quad \begin{pmatrix} \nu_L^e \\ e_L \end{pmatrix} \; ; \quad \begin{pmatrix} \nu_L^\mu \\ \mu_L \end{pmatrix} \; ; \quad \begin{pmatrix} \nu_L^\tau \\ \tau_L \end{pmatrix} \tag{7.6}$$

$$u_{Ri} \; : \; u_R \, ; \; c_R \, ; \; t_R \tag{7.7}$$

$$d_{Ri} \quad : \quad d_R\,;\ s_R\,;\ b_R \tag{7.8}$$

$$e_{Ri} \quad : \quad e_R\,;\ \mu_R\,;\ \tau_R \tag{7.9}$$

where we have introduced a family or generation index $i = 1..3$. The Yukawa couplings become now matrices in family space, a priori completely arbitrary. We have three types of terms

$$(y_e)_{ij}\, v\, \bar{\ell}_{Li}\, \Phi\, e_{Rj} + h.c. \tag{7.10}$$

$$(y_d)_{ij}\, v\, \bar{q}_{Li}\, \Phi\, d_{Rj} + h.c. \tag{7.11}$$

$$(y_u)_{ij}\, v\, \bar{q}_{Li}\, \tilde{\Phi}\, u_{Rj} + h.c. \tag{7.12}$$

Instead, the kinetic terms are written as, for example

$$\bar{e}_{Ri}\gamma^\mu D_\mu e_{Ri} \tag{7.13}$$

interactions with gauge fields do not mix families. Therefore, the expression for the currents 7.3 is trivially extended to the three-families case.

Now, when we want to calculate some physical process, we always refer to states that are mass eigenstates, particles with a definite mass. In the SM with more than one family we have

$$\text{Mass eigenstates} \neq \text{Interaction eigenstates}$$

We will be interested in going to the mass eigenstates basis –and there we will family-changing interactions

7.1 Two families

The world was simple with just two families. Let us call the original fields, interaction eigenstates

$$\ell^0_{1L},\ \ell^0_{2L};\ q^0_{1L},\ q^0_{2L} \tag{7.14}$$

$$e^0_R,\ \mu^0_R;\ d^0_R,\ s^0_R;\ u^0_R,\ c^0_R \tag{7.15}$$

and we introduce the collective notation $\mu_R^0 = e_{2R}^0$, etc. We will have three matrices in family space,

$$M_e^{ij} = y_e^{ij} v \quad M_d^{ij} = y_d^{ij} v \quad M_u^{ij} = y_u^{ij} v \tag{7.16}$$

with $i = 1..2$, complex and arbitrary. Combinations

$$M_f^\dagger M_f \; ; \quad M_f M_f^\dagger \tag{7.17}$$

are two hermitic matrices, different for each $f = e, u, d$. We can diagonalize them with a unitary matrix for each combination

$$D_f^2 = \begin{pmatrix} m_{f1}^2 & \\ & m_{f2}^2 \end{pmatrix} = U_{fL}^\dagger M_f M_f^\dagger U_{fL} = U_{fR}^\dagger M_f^\dagger M_f U_{fR} \tag{7.18}$$

That is to say

$$D_f = \begin{pmatrix} m_{f1} & \\ & m_{f2} \end{pmatrix} = U_{fL}^\dagger M_f U_{fR} \tag{7.19}$$

We need two *different* matrices for each mass matrix. The eigenvectors in the diagonal mass matrix basis are found by transforming independently the left and right states:

$$(f_L)_i = (U_{fL}^\dagger)_{ij} (f_L^0)_j \tag{7.20}$$

$$(f_R)_i = (U_{fR}^\dagger)_{ij} (f_R^0)_j \tag{7.21}$$

We therefore have 6 different unitary matrices. Explicitly, the change of basis is

$$\begin{pmatrix} e \\ \mu \end{pmatrix}_{L,R}^0 = U_{L,R}^e \begin{pmatrix} e \\ \mu \end{pmatrix}_{L,R} \tag{7.22}$$

$$\begin{pmatrix} d \\ s \end{pmatrix}_{L,R}^0 = U_{L,R}^d \begin{pmatrix} d \\ s \end{pmatrix}_{L,R} \tag{7.23}$$

$$\begin{pmatrix} u \\ c \end{pmatrix}_{L,R}^0 = U_{L,R}^u \begin{pmatrix} u \\ c \end{pmatrix}_{L,R} \tag{7.24}$$

How do interactions look like in the mass eigenstates basis? let us first look at the neutral currents, as the electromagnetic one

$$
\begin{aligned}
J_\mu^{em} &= \sum_f \overline{f}^0 \gamma_\mu Q f^0 \\
&= \sum_f \overline{f}^0_L \gamma_\mu Q f^0_L + \overline{f}^0_R \gamma_\mu Q f^0_R \\
&= \sum_f \overline{f}_L U^\dagger_{fL} \gamma_\mu Q U_{fL} f_L + \overline{f}_R U^\dagger_{fR} \gamma_\mu Q U_{fR} f_R \\
&= \sum_f \overline{f} \gamma_\mu Q f
\end{aligned}
\tag{7.25}
$$

This is due to the fact that particles inside each family have different charges. The same happens with the neutral weak current, just replacing Q for $T_{3L} - Q \sin^2 \theta_W$. We see that *neutral currents do not mix families.*

Situation is very different in charged currents that give interactions with $W^\pm_\mu$. First, the quark current

$$
\begin{aligned}
J^+_{\mu q} &= \overline{u}^0_L \gamma^\mu d^0_L + \overline{c}^0_L \gamma^\mu s^0_L \\
&= \begin{pmatrix} \overline{u}^0_L & \overline{c}^0_L \end{pmatrix} \gamma^\mu \begin{pmatrix} d^0_L \\ s^0_L \end{pmatrix} \\
&= \begin{pmatrix} \overline{u}_L & \overline{c}_L \end{pmatrix} U^\dagger_{uL} \gamma^\mu U_{dL} \begin{pmatrix} d_L \\ s_L \end{pmatrix}
\end{aligned}
\tag{7.26}
$$

Defining the unitary matrix

$$
V \equiv U^\dagger_{uL} U_{dL} \tag{7.27}
$$

we can write the interaction in the mass eigenstates basis as

$$
\frac{g}{\sqrt{2}} \left[\begin{pmatrix} \overline{u}_L & \overline{c}_L \end{pmatrix} \gamma^\mu V \begin{pmatrix} d_L \\ s_L \end{pmatrix} \right] W^+_\mu \tag{7.28}
$$

so that charged bosons do mix families.

In the leptonic sector, the reasoning goes along the same lines. The leptonic current is

$$
\begin{aligned}
J^+_{\mu \ell} &= \begin{pmatrix} \overline{\nu}^0_{eL} & \overline{\nu}^0_{\mu L} \end{pmatrix} \gamma^\mu \begin{pmatrix} e^0_L \\ \mu^0_L \end{pmatrix} \\
&= \begin{pmatrix} \overline{\nu}^0_{eL} & \overline{\nu}^0_{\mu L} \end{pmatrix} \gamma^\mu U_{eL} \begin{pmatrix} e_L \\ \mu_L \end{pmatrix}
\end{aligned}
\tag{7.29}
$$

But the situation is very different here, since neutrinos are massless and appear only in interaction terms. We can safely rotate them, defining new fields

$$\left(\begin{array}{c} \nu_{eL} \\ \nu_{\mu L} \end{array} \right) = U^{\dagger}_{eL} \left(\begin{array}{c} \nu^0_{eL} \\ \nu^0_{\mu L} \end{array} \right) \tag{7.30}$$

so that the leptonic current is still diagonal in family space

$$J^{+}_{\mu\ell} = \left(\begin{array}{cc} \overline{\nu}_{eL} & \overline{\nu}_{\mu L} \end{array} \right) \gamma^{\mu} \left(\begin{array}{c} e_L \\ \mu_L \end{array} \right) \tag{7.31}$$

So we have found that the SM for two families precisely reproduces the result we wanted: only quarks get mixed, and only in the charged currents. We can do even better, and rewrite the interactions in terms if the Cabibbo angle defined in Chapter 2. In order to do that, notice that V is unitary and we can therefore write it with four parameters: one rotation angle and three phases

$$V = e^{i\gamma} \left(\begin{array}{cc} \cos\theta e^{i\alpha} & \sin\theta e^{i\beta} \\ -\sin\theta e^{i\beta} & \cos\theta e^{-i\alpha} \end{array} \right) \tag{7.32}$$

The global phase can always be eliminated by redefining $W^{\pm}_{\mu}$. Interactions 7.28 now look like

$$\begin{array}{c} \left[\overline{u}_L \gamma^{\mu} d_L \cos\theta e^{i\alpha} + \overline{u}_L \gamma^{\mu} s_L \sin\theta e^{i\beta} \right. \\ \left. -\overline{c}_L \gamma^{\mu} d_L \sin\theta e^{-i\beta} + \overline{c}_L \gamma^{\mu} c_L \cos\theta e^{-i\alpha} \, \right] W^{+}_{\mu} \end{array} \tag{7.33}$$

But we still have freedom in defining the states. The mass terms

$$m_q \overline{q}_L q_R \tag{7.34}$$

are invariant under rotations of q_R and q_L by the same phase. We have 4 quarks, and since one global phase is unphysical we are left with three to play with. Choosing

$$d' = e^{i\alpha} d; \quad s' = e^{i\beta} s; \quad c' = e^{i(\alpha+\beta)} \tag{7.35}$$

we get

$$\frac{g}{\sqrt{2}} \left[\left(\begin{array}{cc} \overline{u}_L & \overline{c}_L \end{array} \right) \gamma^{\mu} \left(\begin{array}{cc} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{array} \right) \left(\begin{array}{c} d_L \\ s_L \end{array} \right) \right] W^{+}_{\mu} \tag{7.36}$$

Thus we have arrived at

$$\overline{u}\gamma^{\lambda}(1-\gamma_5)d_\theta W^+_\lambda \; ; \quad d_\theta = \cos(\theta_c)d + \sin(\theta_c)s \tag{7.37}$$

which reproduces observations of strange particle decays. It is important to point out that the Cabibbo angle, measured experimentally, is the first hint of the existence of a second family. It implies that the new particles have the same structure inside the second family as in the first one.

7.2 GIM mechanism

The s quark was the first quark of the second family to be discovered. Until 1974, one could have supposed that the $SU(2)$ structure of the quarks was like

$$\begin{pmatrix} u_L \\ d_{L\theta} \end{pmatrix} \; ; \quad s_{L\theta} \tag{7.38}$$

con

$$d_\theta = d\cos\theta_C + s\sin\theta_C \; ; \quad s_\theta = s\cos\theta_C - d\sin\theta_C \tag{7.39}$$

If this were the case, we would have for the neutral currents

$$J^0_\mu = \overline{d}_{L\theta}\gamma_\mu d_{L\theta}(-\frac{1}{2}+\frac{1}{3}\sin^2\theta_W) + s_{L\theta}\gamma_\mu s_{L\theta}(\frac{1}{3}\sin^2\theta_W) \tag{7.40}$$

Or

$$\begin{aligned} J^0_\mu \;\; = \;\; & \overline{d}_L\gamma_\mu d_L(-\frac{1}{2}\cos^2\theta_C + \frac{1}{3}\sin^2\theta_W) + \overline{s}_L\gamma_\mu s_L(-\frac{1}{2}\sin^2\theta_C + \frac{1}{3}\sin^2\theta_W) \\ & -\frac{1}{2}\sin\theta_C\cos\theta_C(\overline{d}_L\gamma_\mu s_L + \overline{s}_L\gamma_\mu d_L) \end{aligned} \tag{7.41}$$

The last term mixes families through a neutral current, and as we said, this is not observed.

In order to forbid this term, it is enough to suppose that s_L is not an $SU(2)$ singlet, but instead it forms a doublet together with another quark c_L

$$\begin{pmatrix} c_L \\ s_{L\theta} \end{pmatrix} \tag{7.42}$$

This was proposed by Glashow, Iliopoulos and Maiani. The so-called GIM mechanism manages to eliminate the undesired term by postulating a new quark. The discovery of this quark was a big success of the model. Furthermore, not only it is possible to predict the c quark, but also its mass, from the measure amplitude of K meson decay.

7.3 Three families

Up ti now, we have the following experimentally tested predictions

1. leptons do not mix with people form other families
2. neutral currents do not mix families
3. charged currents do

We also know that the matrix V is *real*. This means couplings are real, and as a consequence the Lagrangean is CP conserving. Predictions 1), 2) and 3) hold true if we include the third family, because they are a consequence of the $SU(2)$ structure and the third family will preserve it. But now V will not be real, nor will it be possible to parametric it with just one angle, and we will have the very important prediction of CP violation in the SM.

The problem is to determine how many parameters should the matrix V have, a unitary matrix of dimension $N = number of families$. We can think of it as an orthogonal matrix, parametrized by Euler's angles, plus additional phases. We will have in total N^2 parameters, not counting a global phase, therefore

$$N^2 \text{ parameters} = N(N-1)/2 \text{ Euler angles} + N(N+1)/2 \text{ phases.}$$

and we will have $2N$ quarks in total, whose $2N-1$ phases (not counting the global one) we are free to redefine. Then

$$N(N+1)/2 \text{ - } (2N-1) = (N-2)(N-1)/2 \text{ physical phases}$$

For 1 or 2 families, all phases can be removed. For 3 or more, physical phases will appear and give complex couplings and therefore CP violation.

In 1973, Kobayashi and Maskawa proposed to explain observations of CP violation in K mesons decay by postulating a third family, today fully discovered. The matrix V is therefore called V_{CKM}: Cabibbo-Kobayashi-Maskawa. A usual parametrization is

$$V_{CKM} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ -s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}s_{13} \end{pmatrix} \tag{7.43}$$

where

$$c_{ij} = \cos\theta_{ij}\ ; s_{ij} = \sin\theta_{ij} \tag{7.44}$$

and θ_{ij} are the 3 Euler angles.

Box 7.1: CP violation in $K\overline{K}$

The neutral meson K^0 is a state formed by $\overline{d}s$. Diagrams as in (7.1) give an effective hamiltonian for the states K^0 and $\overline{K}^0$, which is O-conjugated state. Its matrix elements

$$\langle\alpha'|H_{\text{eff}}|\alpha\rangle \tag{7.45}$$

will be real in the case of two generations, where as we seen, interactions do not involve physical phases. The effective Hamiltonian preserves CP and contains the mass terms

$$m_K(K^0K^0 + \overline{K}^0\overline{K}^0) + 2\delta m_K K^0\overline{K}^0 \tag{7.46}$$

Diagonalizing

$$\begin{pmatrix} m_K & \delta m_K \\ \delta m_K & m_K \end{pmatrix} \tag{7.47}$$

we get the eigenstates

$$K_1 = \frac{1}{\sqrt{2}}(K^0 + \overline{K}^0)\ ; \quad K_2 = \frac{1}{\sqrt{2}}(K^0 - \overline{K}^0) \tag{7.48}$$

with masses

$$m_1 = m_K + \delta m_K\ ; \quad m_2 = m_K - \delta m_K \tag{7.49}$$

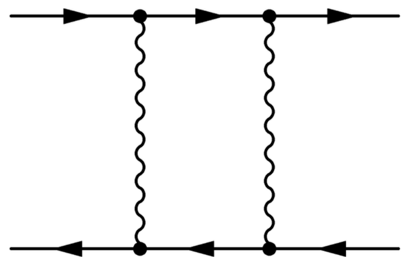


Figure 7.1: 1-loop diagram for the calculation of the effective Hamiltonian of the $K^0, \overline{K}^0$ system

respectively. In the two-generation case, quarks running in the loop in 7.1 are u and c. One can then calculate the mass difference

$$\frac{\delta m_K}{m_K} = \frac{g^4}{16\pi^2}\left(\sin 2\theta \frac{m_c^2 - m_u^2}{M_W^2}\right)\left(\frac{m_K}{M_W}\right)^2 \tag{7.50}$$

This is very small. With

$$m_c \sim 1.5GeV; \quad m_u \sim 10MeV \tag{7.51}$$

one gets

$$\frac{\delta m_K}{m_K} \simeq 10^{-13} \tag{7.52}$$

which agrees with experiment.

As we know however, there are 3 generations and the Lagrangean is not CP conserving. The diagram that gives the effective matrix element $\langle K^0|H_{\rm eff}|\overline{K}^0\rangle$ *will be proportional to*

$$V_{ts}^2 V_{td}^{*2} \tag{7.53}$$

while the one giving $\langle \overline{K}^0 | H_{\text{eff}} | K^0 \rangle$ *will be proportional to*

$$V_{td}^2 V_{ts}^{*2} \tag{7.54}$$

which is different, as it involves the phase δ.

The mass matrix in this case looks like

$$\begin{pmatrix} m_K & \delta m_K (1+\epsilon) \\ \delta m_K (1-\epsilon) & m_K \end{pmatrix} \tag{7.55}$$

And the mass eigenstates

$$K_S = K_2 + \epsilon K_1 \; ; \quad K_L = K_1 + \epsilon K_2 \tag{7.56}$$

are not CP eigenstates anymore. The parameter ϵ *measures CP violation in this system, and can be written in terms of the* V_{CKM} *matrix elements as*

$$\epsilon = \sin\theta_{12} \sin\theta_{13} \sin\theta_{23} \delta \tag{7.57}$$

This is very small, approximately $\epsilon \sim 10^{-3}$.

Exercise 14
Show that V_{CKM} in the parametrization (7.43) is unitary. Show that the phase δ implies CP violation

Exercise 15
Suppose that the masses of one of the species of quarks (up or down) are degenerate. Show that V_{CKM} has no physical meaning (it is proportional to the identity matrix)

Chapter 8

Neutrinos

By construction the SM has no neutrino mass term. In the absence of ν_R, all such terms are forbidden by the symmetry. This of course is due to the fact that there was no evidence of a neutrino mass until very recently. However, ever since the existence of a neutrino was first postulated, the possibility of a very small neutrino mass has always been in the back of physicist's minds, and in fact experiments could only give us an upper limit.

As we have seen in the last chapter, the absence of mass terms for neutrinos implies that there is no family mixing among leptons, in contrast with the situation in the quark sector. This suggests an indirect manner of detecting neutrino mass: one could look for oscillations among neutrino species. Let us see a simple example.

8.1 Oscillations

Let us call the three neutrino interaction eigenstates

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} \tag{8.1}$$

and the mass ones

$$\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \tag{8.2}$$

They will be connected by a matrix similar to V_{CKM}

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = V_\nu \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \tag{8.3}$$

For the time being, let us say that V_ν takes the form (7.43). Let us suppose that we have an interaction producing a ν_e neutrino beam at $t = 0$. The $|\nu_e\rangle$ state will be a linear combination of mass eigenstates

$$|\nu_e(0)\rangle = a\,|\nu_1\rangle + b\,|\nu_2\rangle + c\,|\nu_3\rangle \tag{8.4}$$

If this neutrino propagates freely, at a later time t we will have

$$|\nu_e(t)\rangle = a\,e^{-iE_1 t}|\nu_1\rangle + b\,e^{-iE_2 t}|\nu_2\rangle + c\,e^{-iE_3 t}|\nu_3\rangle \tag{8.5}$$

where

$$E_i^2 = p^2 + m_i^2 \tag{8.6}$$

and the masses m_i are in principle different. We can calculate the probability of detecting an " interaction" neutrino ν_α at t

$$\mathcal{P}(\nu_\alpha, t) = |\langle\nu_\alpha|\nu_e(t)\rangle|^2 \tag{8.7}$$

and this will give us the oscillation probability of ν_e into ν_α. For instance, the probability of measuring ν_e in this beam can be written as

$$\begin{aligned} \mathcal{P}(\nu_e \to \nu_e(t)) &= 1 - A\,[1 - \cos(E_1 - E_2)t] \\ &+ B\,[1 - \cos(E_1 - E_3)t] + C\,[1 - \cos(E_2 - E_3)t] \end{aligned} \tag{8.8}$$

where coefficients A, B, C are given in terms of the elements of V_ν. Taking the relativistic limit $p \gg m_i$

$$E_i \simeq p + \frac{m_i^2}{2p} \tag{8.9}$$

we will have

$$E_i - E_j = \frac{m_i^2 - m_j^2}{2p} \equiv \frac{\Delta m_{ij}^2}{2p} \tag{8.10}$$

That is, oscillation probabilities depend upon the mass differences.

Taking neutrinos propagating at he speed of light, $c = 1$, we can write the oscillation probability at a distance x from the source Defining the *oscillation length*

$$\ell_{ij} = \frac{2\pi}{E_i - E_j} \simeq \frac{4\pi p}{\Delta m_{ij}^2} \tag{8.11}$$

we have

$$(E_i - E_j)t = \left(\frac{2\pi x}{\ell_{ij}}\right) \tag{8.12}$$

In terms of the elements of V_ν,

$$\mathcal{P}_{\nu_\alpha \to \nu_\beta} = \sum_i |V_{\alpha i}|^2 |V_{\beta i}|^2 + \sum_{i \neq j} V_{\alpha i} V_{\beta i}^* V_{\alpha j}^* V_{\beta j} \cos\left(\frac{2\pi x}{\ell_{ij}}\right) \tag{8.13}$$

Oscillations of neutrinos produced in the sun (solar neutrino problem) have been confirmed to happen in the last few years, as have oscillation of atmospheric neutrinos. The precise determination of the mixing angles and mass differences is at the moment subject of intense research. At the moment of writing this, observations are consistent with a fit

$$\Delta m_{12}^2 \sim 10^{-4} eV^2 \; ; \quad \Delta m_{23}^2 \sim 10^{-3} eV^2 \tag{8.14}$$

for the masses and

$$\theta_{12} \sim 30^o \; ; \quad \theta_{23} \sim 45^o \tag{8.15}$$

for the mixing angles.

8.2 Mass for the neutrinos

We have seen that for a neutral fermion, two mass terms can be written: Dirac and Majorana,

$$m_D \overline{\psi} \psi \; ; \quad m_M \psi^T C \psi \tag{8.16}$$

The absence of ν_R prevents us from writing a Dirac mass term for the neutrinos. Now, ν_L is in fact a singlet under the low energy symmetry group $SU(3) \times U(1)_{em}$, so that a Majorana mass term is in principle possible. But we have built the model supposing that the underlying theory is $SU(2)_L \times U(1)_Y$ *spontaneously* broken, and this Majorana term would break it explicitly. If we do not want to spoil this construction, the only possibility is introducing the missing state, ν_R.

These new states ν_R (say one per family), would be really neutral under $SU(3)_c \times SU(2)_L \times U(1)_Y$. We can then write

$$M_R \nu_R^T C \nu_R \tag{8.17}$$

and a Yukawa interaction with the Higgs, just as we did for the charged particles, giving a Dirac mass

$$y_{ij}^{\nu} \overline{\nu}_{Li} \tilde{\Phi} \nu_{Rj} \Rightarrow +h.c. m_D \overline{\nu}_L \nu_R + h.c. \tag{8.18}$$

This means that ν_L and ν_R are not mass eigenstates, even in the one family case, where we would have

$$m_D \overline{\nu}_L \nu_R + M_R \overline{\nu}_R^c \nu_R \tag{8.19}$$

The neutrino mass matrix looks like

$$\begin{pmatrix} 0 & m_D \\ m_D & M_R \end{pmatrix} \tag{8.20}$$

and its eigenvalues are

$$m_{\pm} = \frac{M_R}{2} \pm \frac{M_R}{2} \sqrt{1 + \frac{m_D^2}{M_R^2}} \tag{8.21}$$

See-saw

The interesting point is that M_R M scale (Φ vev), and can in principle take any value. In particular, one could have

$$M_R \gg m_D \tag{8.22}$$

This would be a natural situation if, for example, M_R comes form a spontaneous breakdown of parity at a higher scale. In this case we would have from (8.21) a heavy state with

$$m_{\nu_+} \sim M_R \tag{8.23}$$

consisting essentially of ν_R with little admixture of ν_L, and a light state

$$m_{\nu_-} \sim \frac{m_D^2}{M_R} \tag{8.24}$$

made up almost entirely of ν_L. This is the famous *see-saw* mechanism: a right-handed neutrino, neutral under the SM group and with a large Majorana mass, forces the left-handed neutrino to be very light in comparison with the charged fermions. Notice this can only happen to neutrinos: a particle that is almost neutral can easily be light. The see-saw mechanism is particularly interesting if considered in the context of Left-Right theories, where all the right-handed particles of teh SM transform under a $SU(2)_R$ group, mirroring their left-handed partners. If this $SU(2)_R$ is spontaneously broken at a high scale, condition (8.22) comes naturally.

Three families

If we include one right-handed neutrino in each family, we will have a matrix analogous to V_{CKM}

$$V_\nu = U_L^{\nu\dagger} U_L^e \tag{8.25}$$

All the discussion we made for the quark sector can be repeated here: V_ν is a unitary matrix, and for N families it has N^2 real parameters. $N(N-1)/2$ of them will be angles and $N(N+1)/2$ will be phases. In order to rotate these away, we simultaneously redefine ν_{Ri} and ν_{Li}. However, we now do not have $2N-1$ phases at our disposal, as we did for the quarks. The term

$$\nu_R^T C \nu_R \tag{8.26}$$

is not invariant under phase changes of ν_R. This means that neutrinos cannot absorb phases, only electrons can. We can eliminate only N of them and we will have

$$\frac{N(N-1)}{2} \quad \text{phases} \tag{8.27}$$

in the leptonic sector. This implies CP violation even in the two-families case. For three families we will have three phases

- one “ Dirac”, phase, as in the quarks
- two “ Majorana” phases

giving additional contributions to CP violation.

V_ν is usually parametrized as

$$V_\nu = V \begin{pmatrix} m_1 e^{i\rho} & & \\ & m_2 & \\ & & m_3 e^{i\sigma} \end{pmatrix} V^\dagger \tag{8.28}$$

where V is written as V_{CKM}.

Exercise 16

Find coefficients A,B and C in (8.8)

Exercise 17

Suppose there exist three ν_{Ri}, and the $SU(2)_L$ structure is repeated for the right-handed particles, which will then transform under the fundamental representation of another group, $SU(2)_R$. The full symmetry group will be the Left-Right group, $SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_a$. Find the charge of each particle under $U(1)_a$, and give the hypercharge as a function of this new charge and of T_{3R}. Write the new expression for the electromagnetic charge