Neutrino GUT Course

Lecture $\overline{\underline{XXV}}$

7/2/2023

LMU

Winter 2023

SO(2N) algebras

(Spin(2N) —"—)

- $OO^T = O^T O = 1$
- $\det O = 1$

$$\boxed{\theta_{ij} = -\theta_{ji}}$$

$$O = e^{i\,\theta_{ij}\,L_{ij}}$$

$i,j = 1, \dots, 2N$

$$(L_{ij})_{kl} = -i\left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right)$$

$$[L_{ij}, L_{kl}] = i\left(l_{ij}\,\delta_{jl} - \delta_{il}\,\delta_{jk}\right)$$

$$L_{12} = \begin{pmatrix} 0 & -i & \cdots \\ i & 0 & \cdots \\ \vdots & \cdots & \end{pmatrix}$$

$$L_{34} = \begin{pmatrix} \times & \cdots & & \\ \vdots & & 0 & -i \\ & & i & 0 \\ & & & \end{pmatrix}$$

$$\{L_{12}, L_{34}, \ldots, L_{2N-1,2N}\} =$$

$$= \text{Cartan sub-algebra}$$

$\boxed{\text{mutually commuting}}$

$$\boxed{r(SO(2N)) = N}$$

$$\boxed{L_{12}, L_{34}, \ldots : \text{ eigenvalues } \pm 1}$$

Spinors

$$\{\Gamma_i, \Gamma_j\} = 2\,\delta_{ij} \quad \text{Clifford}$$

$$\Sigma_{ij} = \frac{1}{4i}[\Gamma_i, \Gamma_j]$$

$$[\Sigma_{ij}, \Sigma_{kl}] = i\,(\Sigma_{ik}\,\delta_{jl} - \Sigma_{il}\,\delta_{jk})$$

$$S = e^{i\theta_{ij}\Sigma_{ij}}$$

$$\psi \longrightarrow S\psi \quad (\psi = \text{spinor})$$

$$\boxed{\Gamma_{FIVE} = (-i)^N\,\Gamma_1 \cdots \Gamma_{2N}}$$

$$\therefore\ \Gamma_{FIVE}^{\,2} = 1$$

$$\{\Gamma_{FIVE}, \Gamma_i\} = 0$$

$$[\Gamma_{FIVE}, \Sigma_{ij}] = 0$$

$$P_{+,-} = \frac{1 \pm \Gamma_{FIVE}}{2}$$

$$\boxed{\Psi_+ = P_+ \Psi} \qquad (\Psi_- = P_- \Psi)$$

Irreducible spinor of $SO(2N)$

$\Longleftrightarrow$ Ψ_L of Lorentz

$$\boxed{\begin{array}{ccc} SO(2) & \longrightarrow & Spin(2) \\ \parallel & & \\ U(1) & & \end{array}}$$

Simplest example

$$\Gamma_1 = \sigma_1, \quad \Gamma_2 = \sigma_2$$

$$\Sigma \equiv \Sigma_{12} = \frac{1}{4i}[\sigma_1, \sigma_2] = \frac{\sigma_3}{2}$$

only generator

$$\Gamma_{FIVE} = (-i)\,\Gamma_1\Gamma_2 = -i\,\sigma_1\sigma_2$$

$$\boxed{\Gamma_{FIVE} = \sigma_3}$$

$$S = e^{i\theta\Sigma}$$

$$\psi \to e^{i\theta\,\sigma_3/2}\,\psi$$

$$\psi = \begin{pmatrix} u \\ d \end{pmatrix} \Rightarrow \psi_+ = \begin{pmatrix} u \\ 0 \end{pmatrix}$$

$$\boxed{u \to e^{i\theta/2} u}, \quad d \to e^{-i\theta/2} d$$

physical spinor

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \to e^{i\theta L_{12}} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

$$L_{12} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\{\Sigma_{12}, \Sigma_{34}, \ldots, \Sigma_{2N-1,2N}\} = \text{Cartan}$$

$$\Sigma_{12} = \frac{1}{4i}[\Gamma_1, \Gamma_2] = \frac{1}{2i}\Gamma_1\Gamma_2$$

eigenvalues: $\pm 1/2$

$$\boxed{\begin{aligned} \phi(2\pi) &= \phi(0) \\ \psi(2\pi) &= -\psi(0) \end{aligned}}$$ $\leftarrow$ spinors

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digression

- $e^{i\Theta L_{12}} = e^{i\Theta\sigma_2} = \cos\Theta + i\sin\Theta\,\sigma_2$

$$= \begin{pmatrix} \cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta \end{pmatrix}$$

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \to \begin{pmatrix} c & s \\ -s & c\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$$

$$\phi_\pm = (\phi_1 \pm i\,\phi_2)\frac{1}{2}$$

$$\boxed{\phi_\pm \overset{?}{\longrightarrow} e^{\pm i\theta}\,\phi_\pm} \quad \underline{\overline{\text{check!}}}$$

- Lorentz: charge conjugation C

$\therefore\ \psi_L^T C\, \psi_L = \text{inv.}$

$\Downarrow$

by analogy: in $SO(2)$

"charge conjugation" B

$$\therefore \quad \psi^T B \psi = i w_0$$

where $\psi \to e^{i\theta \sigma_3/2} \psi$

$$\left(\psi^T \psi \neq i w_0 \right)$$

$$\hookrightarrow \psi^T e^{i\sigma_3\theta/2} e^{i\sigma_3\theta/2} \psi$$

$$= \psi^T e^{i\sigma_3\theta} \psi \neq i w_0$$

need : $\{\sigma_3, B\} = 0$

$$\psi^T B \psi \to \psi^T e^{i\theta/2\,\sigma_3} B\, e^{i\sigma_3/2\,\theta} \psi$$

$$= \psi^T B \underbrace{e^{-i\theta/2\,\sigma_3} e^{i\theta/2\,\sigma_3}}_{\mathbb{1}} \psi$$

$$= \psi^T B \psi_V \quad \text{(invariant)}$$

$$\Downarrow$$

$$\boxed{B = \sigma_1 \; (\sigma_2)}$$

- $B = \sigma_1$

$$\Rightarrow \quad \psi^T B \psi = (u \; d) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}$$

$= \cancel{2ud}$

since, only $\psi_+ = \begin{pmatrix} u \\ 0 \end{pmatrix}$

$\Downarrow$

$SO(2) =$ chiral

$\Longleftrightarrow$ <u>no</u> direct mass term

$SO(2) \longrightarrow SO(3)$

$\Updownarrow$

$Spin(3) = SU(2)$

$T_1 = \sigma_1, \quad T_2 = \sigma_2, \quad T_3 = \sigma_3$

$$\Downarrow$$

$$\Sigma_{ij} = \frac{1}{4i}\,[\Gamma_i, \Gamma_j] = \frac{1}{4i}(2i\,\varepsilon_{ijk}\,\sigma_k)$$

$$\Sigma_{ij} = \frac{1}{2}\,\varepsilon_{ijk}\,\sigma_k$$

$$\Sigma_{ij} = \varepsilon_{ijk}\,T_k$$

$$\Longleftrightarrow\ [T_i, T_j] = i\,\varepsilon_{ijk}\,T_k$$

$$T_i \equiv \sigma_i/2$$

<u>no</u> Γ_{FIVE}

$\Rightarrow$ <u>no</u> chirality

- Lorentz : $\psi_L^T C \psi_L = \text{inv.}$

  opposite from $SO(2)$!!

$$\Updownarrow$$

$SO(4) =$ like Lorentz

$SO(2) =$ good (chiral)

$SO(4) =$ bad (vector-like)

$$\Updownarrow$$

$$\boxed{SO(4N+2) = \text{good}}$$

$SO(4N) =$ bad

- chirality of $SO(2)$:

$$u \to e^{i\theta/2} u$$

$$\Rightarrow \boxed{u u \neq \text{invariant}}$$

$\Leftarrow$ <u>no</u> mass term

$$SO(4) = SU(2)_L \times SU(2)_R$$

$SO(2)$: $\quad \Gamma_1 = \sigma_1, \quad \Gamma_2 = \sigma_2$

$SO(3)$: $\quad \Gamma_1 = \sigma_1, \quad \Gamma_2 = \sigma_2, \quad \Gamma_3 = \sigma_3$

$$\downarrow$$

$SO(4)$: $\quad \Gamma_a = \begin{pmatrix} 0 & \Gamma_a \\ \Gamma_a & 0 \end{pmatrix} \quad a = 1, 2, 3$

$$\Leftrightarrow \quad \Gamma_a = \begin{pmatrix} 0 & \sigma_a \\ \sigma_a & 0 \end{pmatrix}$$

$$\Gamma_4 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\Rightarrow \{ \Gamma_a, \Gamma_4 \} = 0 \quad , \quad \Gamma_4^2 = 1$$

$$\Downarrow$$

$$\{\Gamma_i, \Gamma_j\} = 2\delta_{ij} \qquad i = a, 4$$

$$\Sigma_{ab} = \frac{1}{4i}[\Gamma_a, \Gamma_b] =$$

$$= \frac{1}{4i} 2 \varepsilon_{abc} i \begin{pmatrix} \sigma_c & 0 \\ 0 & \sigma_c \end{pmatrix}$$

$$= \frac{1}{2} \varepsilon_{abc} \sigma_c \mathbb{1} \qquad (3)$$

$$\Sigma_{a4} = \frac{1}{2i} \Gamma_a \Gamma_4 = \frac{i}{2i} \begin{pmatrix} \sigma_a & 0 \\ 0 & -\sigma_a \end{pmatrix}$$

$$\boxed{\Sigma_{a4} = \frac{1}{2} \begin{pmatrix} \sigma_a & 0 \\ 0 & -\sigma_a \end{pmatrix}}$$

$$\Sigma_{ab} = \varepsilon_{abc} T_c$$

$$T_c = \frac{1}{2}\begin{pmatrix} \sigma_c & 0 \\ 0 & \sigma_c \end{pmatrix}$$

$$\Sigma_{a4} = T_a{}'$$

$$T_a{}' = \frac{1}{2}\begin{pmatrix} \sigma_a & 0 \\ 0 & -\sigma_a \end{pmatrix}$$

$$\Downarrow$$

$$\frac{T_a + T_a{}'}{2} = \frac{1}{2}\begin{pmatrix} \sigma_a & 0 \\ 0 & 0 \end{pmatrix} \equiv T_a{}^L$$

$$\frac{T_a - T_a{}'}{2} = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \sigma_a \end{pmatrix} \equiv T_a{}^R$$

$$\Downarrow$$

$$\boxed{SO(4) = SU(2)_L \times SU(2)_R}$$

$$\Gamma_{FIVE} = (-i)^2 \underbrace{\Gamma_1 \Gamma_2}_{\downarrow} \Gamma_3 \Gamma_4$$

$$= (-1) \begin{pmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix} \begin{pmatrix} i\sigma_3 & 0 \\ 0 & -i\sigma_3 \end{pmatrix}$$

$$= (-1) \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\psi = \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix}, \quad \psi_+ = \frac{1 + \Gamma_{FIVE}}{2} \psi$$

$$\Rightarrow \psi_+ = \begin{pmatrix} \psi_u \\ 0 \end{pmatrix}$$

- $\Psi^T B \Psi \overset{?}{=}$ invariant

analogy: $C = i\gamma_2\gamma_0$

$$B \overset{!}{=} \Gamma_2 \Gamma_4$$

$$\left(B \overset{!}{=} \Gamma_1 \Gamma_3 \right)$$

$$\Gamma_2 \Gamma_4 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}$$

$$\psi_u \to e^{i\theta_a \sigma_a/2} \psi_u \qquad a = 1,2,3$$

$$\Psi^T B \Psi = \Psi^T \Gamma_2 \Gamma_4 \Psi$$

$$= \underbrace{\psi_u^T \, i\sigma_2 \, \psi_u}_{\text{inv.}} + \cancel{(u \to d)}$$

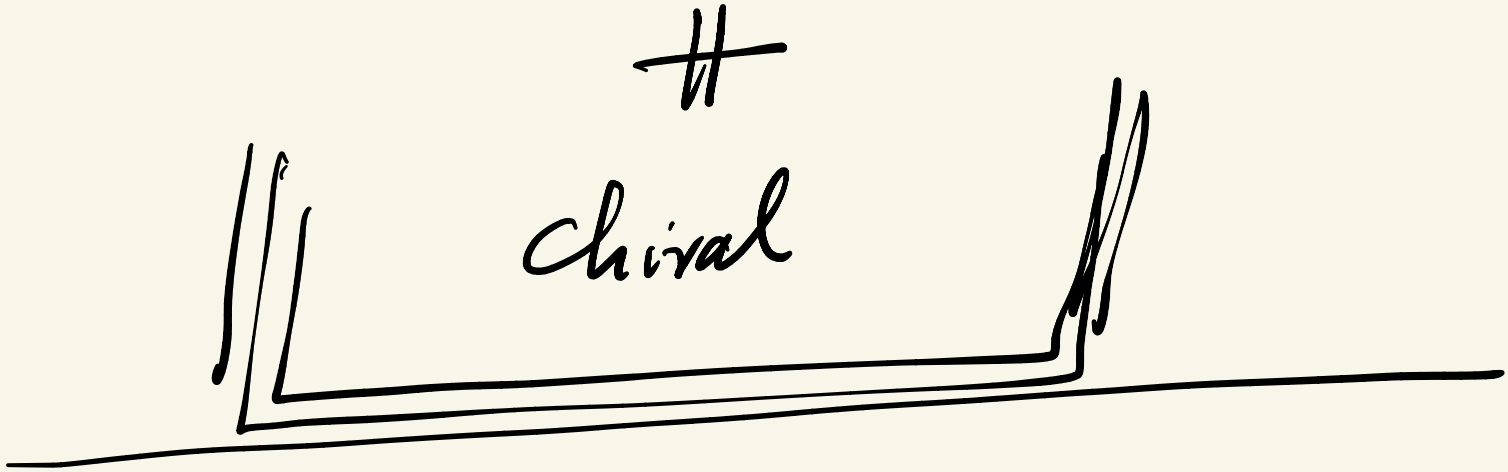
$SO(5)$ $\qquad \Gamma^{(5)}_\alpha = \Gamma_\alpha(SO(4))$

$\alpha = 1, 2, 3, 4$

$$\Gamma^{(5)}_5 = \Gamma_{FIVE}$$

$$\Rightarrow \{\Gamma^{(5)}_i, \Gamma^{(5)}_j\} = 2\delta_{ij} \qquad i = \alpha, 5$$

$$\boxed{SO(2N+1) = \text{bad}}$$



$SO(6)$ $\qquad a = 1, 2, \ldots, 5$

$$\Gamma_a^{(6)} = \begin{pmatrix} 0 & \Gamma_a^{(5)} \\ \Gamma_a^{(5)} & 0 \end{pmatrix}$$

$$\Gamma_6^{(6)} = \begin{pmatrix} 0 & -i\mathbb{1}_4 \\ i\mathbb{1}_4 & 0 \end{pmatrix}$$

$$\Gamma_{FIVE} = (-i)^3\, \Gamma_1^{(6)} \cdot \Gamma_2^{(6)} \cdots \Gamma_6^{(6)}$$

$$\Downarrow$$

$$\Gamma_{FIVE} \overset{?}{=} \begin{pmatrix} \mathbb{1}_4 & 0 \\ 0 & -\mathbb{1}_4 \end{pmatrix}$$

$$\psi = \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix}$$

$$\Rightarrow \boxed{\psi_+ = \begin{pmatrix} \psi_u \\ 0 \end{pmatrix} \quad \text{PROVE!}}$$

- $SO(6) =$ good?

$$\psi^T B \psi$$

$$B = \Gamma_2 \Gamma_4 \Gamma_6 \quad \text{(Check)}$$
$$= \text{off-diagonal}$$

$$\Psi^T B \Psi \propto \Psi_u^T \,{-}{-}\, \Psi_d$$

but we take only Ψ_u

$\Rightarrow$ $\boxed{\text{NO } \Psi_u \Psi_u \,!}$

$\Rightarrow$ $SO(6) \sim SO(2)$

$SO(8) \sim SO(4)$

$\Downarrow$

$SO(4N) \leftarrow$ direct mass term

$=$ bad

$\Downarrow$

$\boxed{\text{SO}(4N+2)}$ $\boxed{\text{to be realistic}}$

~~$SO(8)$ $\psi = 16$ dim.~~

~~$\Rightarrow$ $\psi_+ = 8$ dim.~~

bad

$\underline{SO(9)}$ bad

$\underline{\underline{SO(10)}}$ $2^5 = 32$ cmp. ψ

$\Rightarrow$ $\psi_+ = 16$-dim.

$$16 = \underbrace{10 + \bar{5}}_{SU(5)} + \underbrace{1}_{N = RH}$$

neutrino