

GUT Course 22/23

Lecture XIII

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6/12/2022

LMU

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Fall 2022

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# SU(5) GUT (2)

$$G_{SM} = SU(3)_c \times SU(2)_L \times U(1)_Y$$

$$G_{SM} \subseteq G_{GUT} = \underline{\text{grand}}$$

unification

- $G_{GUT}^{min} = SU(5)$
- $\gamma(G_{SM}) = \gamma(SU(5)) = 4$
- $G_{SM} = H_{max} \quad (H \subseteq SU(5))$   
     $\uparrow$   
    NO larger  $H \supseteq H_{max}$

- another  $H_{max} = SU(4) \times U(1)$

- Cartan =  $\left\{ \underbrace{T_{3c}, T_{8c}}_{SU(3)_c}, \underbrace{T_{3w}}_{SU(2)_L}, Y \right\}$   $U(1)$

$$T_{3c} = \text{diag } \frac{1}{2} \left( \underbrace{1, -1, 0}_{\text{color}}; 0, 0 \right)$$

$$T_{8c} = \text{diag } \frac{1}{2\sqrt{3}} (1, 1, -2; 0, 0)$$

$$T_{3w} = \text{diag } \frac{1}{2} (0, 0, 0; 1, -1)$$

$$Y^{\text{norm}} = \text{diag } N (1, 1, 1; -\frac{3}{2}, -\frac{3}{2})$$

$SU(5)$

$$\underline{5} \longrightarrow U \underline{5} \quad \dots$$

$$UU^\dagger = U^\dagger U = 1, \text{ let } U = 1$$

$$\Rightarrow U = e^{iH} = e^{i\theta_i T_i}$$

$$H^\dagger = H$$

$$i = 1, \dots, 24$$

$$T_i H = 0$$

$$T_i^\dagger = T_i, T_i T_i = 0$$

$$T_i T_j = \frac{1}{2} \delta_{ij}$$

$$[T_i, T_j] = i f_{ijk} T_k$$

$$24 = 20 + 4$$

$\sigma_{1,2}$  all  
over

Cartan

- $T_i \leftrightarrow A_i$

$$D_\mu \psi = (\partial_\mu - ig T_i A_\mu^i) \psi$$

$$24 = 12 + 12$$



new

8 gluons,  $w^+$ ,  $w^-$ ,  $z$ ,  $A = 504$

- Matter = fermions

$$\left. \begin{array}{l} \alpha = 1, 2, 3 \\ = x, y, b \end{array} \right\} \begin{array}{l} \left( \begin{array}{c} u \\ d \end{array} \right)_L^\alpha; \quad (u^c)_L, (d^c)_L \\ \left( \begin{array}{c} \nu \\ e \end{array} \right)_L; \quad (e^c)_L \end{array}$$

$$\bar{5}_F = \begin{pmatrix} d^c \\ d^c \\ d^c \\ l \\ l \end{pmatrix} \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} \begin{array}{l} SU(3)_c \\ \\ \\ \end{array} \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} SU(2)_w$$

$$10_F = \begin{pmatrix} 0 & u^c & u^c & u & d' \\ & 0 & u^c & u & d \\ & & & 0 & u & d \\ & & & & 0 & e^c \\ & & & & & 0 \end{pmatrix}_L$$

As

$$5 \times 5 = 15 + 10$$

(s) sym

antisym (As)

$$\{ 5_i 5_j \} = A_{ij}$$

$$\Rightarrow \boxed{A \rightarrow UAU^T}$$

$$\Leftrightarrow A_{ij} \rightarrow U_{in} A_{ne} U_{ej}^T \\ = U_{in} U_{je} A_{ne}$$

$$\Downarrow \\ \boxed{\hat{T}A = TA + UT^T}$$
$$\Downarrow$$

$$\hat{Q}A = QA + AQ$$

$$\Rightarrow \boxed{(\hat{Q}A)_{ij} = (q_i + q_j) A_{ij}}$$

$$\bar{S}_F = \begin{pmatrix} d^c \\ \dots \\ e \end{pmatrix}_L \quad (\bar{S}_F = S_F^c)$$

$$\Leftrightarrow S_F = \begin{pmatrix} d \\ \dots \\ e^c \\ \nu^c \end{pmatrix}_R \quad \begin{aligned} (e^c)_R &\equiv c \bar{e}_L^T \\ (\nu^c)_R &\equiv c \bar{\nu}_L^T \end{aligned}$$

$$Q \equiv Q_{em}$$

⇓

$$Q = \text{diag} \left( -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}; 1, 0 \right)$$

$$T_{3W} = \text{diag} \left( 0, 0, 0; 1, -1 \right)$$

$$\frac{Y}{2} = \text{diag} \left( -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}; \frac{1}{2}, \frac{1}{2} \right)$$



# Charge quantized

$5_F :$

$$3 Q(d) - Q(e) - Q(\nu) = 0$$
$$\uparrow$$
$$T_3 Q = 0$$

$$\Rightarrow Q(e) + Q(\nu) = 3 Q(d)$$

$$Q(\nu) = Q(e) + 1$$

$$Q = T_3 + \frac{1}{2}$$

$$\Rightarrow Q(u) - Q(d) = 1$$

$S_F$  $\Rightarrow$ 

$$2Q(e) + 1 = 3Q(d)$$

 $T_1 Q = 0$ 

$$10_F \Rightarrow Q(u) = 2Q(d)$$

 $\parallel$ 

$$-Q(u) = -(Q(d) + 1)$$

$$\Rightarrow 3Q(d) = -1$$

$$\Rightarrow Q(d) = -1/3$$

$$Q(e) = -1$$

$$Q(v) = 0$$

$$Q(u) = 2/3$$

$$\Leftrightarrow \text{Tr } Q(10_F) = 0$$



$$3 \left( \cancel{Q(u^c) + Q(u) + Q(d)} \right) + Q(e^c) = 0$$

$$\left[ 3Q(d) = -Q(e^c) = Q(e) \right]$$

Charge is quantized

in  $SU(5)$

$\Rightarrow$  magnetic monopole!

weak mixing angle  
( $\theta_w$ )

$$\begin{aligned}
 \text{SM: } D_\mu &= \dots - ig T_i A_\mu^i - ig' \frac{Y}{2} B \\
 &= \dots - i (g T_3 A_3 + g' \frac{Y}{2} B)_\mu
 \end{aligned}$$

$$\tan \theta_w = g' / g$$

$$T_1 T_3^2 = \frac{1}{2}$$

normalized

↓ gen.

$$\text{SU}(5): \quad g' \frac{Y}{2} = g_1 I_1$$

$$g_1 = g_2 = g_3 \equiv g \equiv g_{\text{U}}$$

unification  $\equiv g_{\text{AUT}}$

iff:  $T_1 I_1^2 = \frac{1}{2}$

but:

$$T_1 \left(\frac{Y}{2}\right)^2 = \frac{1}{9} \cdot 3 + \frac{1}{4} \cdot 2 = \frac{1}{3} + \frac{1}{2}$$

$$= \frac{5}{6} = \frac{5}{3} \cdot \frac{1}{2}$$

$$\frac{Y}{2} = \sqrt{\frac{5}{3}} I_1$$

$$\therefore T_1 \left(\frac{Y}{2}\right)^2 = \frac{5}{3} T_1 I_1^2 =$$

$$= \frac{5}{3} \cdot \frac{1}{2} = \frac{5}{6}$$

$$\text{but: } g_1 I_1 = g' \frac{4}{2}$$

$$\Rightarrow g' = \sqrt{3/5} g_1 = \sqrt{3/5} g$$

$$g_2 = g$$

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$$\tan^2 \theta_w = \left( \frac{g'}{g_2} \right)^2 = \frac{3}{5}$$



$$\sin^2 \theta_w = \frac{3}{8}$$

$$\cos^2 \theta_w = \frac{5}{8}$$

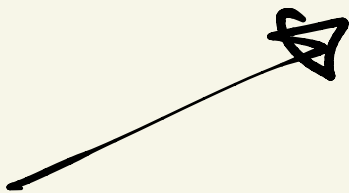


WRONG ???!!

NO

- $g_2 = g_3 = g_1 = g$

at  $M_{GUT}$



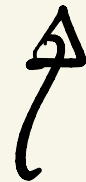
unification scale



$$\sin^2 \Theta_w (M_{\text{GUT}}) = 3/8$$



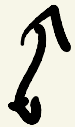
$$\sin^2 \Theta_w (M_w) = 3/8 \pm \dots$$



"running"



$$\alpha = \alpha(E)$$



$$\frac{1}{\alpha(E_2)} = \frac{1}{\alpha(E_1)} + \frac{b}{2\pi} \ln \frac{E_2}{E_1}$$

"crawling" :)



$$(\underline{5} \times \underline{5}) = (\underline{5} \times \underline{5})_s + (\underline{5} \times \underline{5})_{As}$$

$$= 15 + 10$$



all SM fermions  
 2u 15<sub>F</sub>?

Inspection :

①

②

$$5 = (3_c, 1_w) + (1_c, 2_w)$$

$$\Rightarrow 5 \times 5 = \underbrace{\quad} \times \underbrace{\quad}$$

$$\begin{aligned}
&= \textcircled{1} \times \textcircled{2} + \textcircled{2} \times \textcircled{2} + \\
&\quad + \textcircled{1} \times \textcircled{2} + \textcircled{2} \times \textcircled{1} \\
&= (3_c \times 3_c, 1_w) + (1_c, 2_w \times 2_w) \\
&\quad + (3_c, 2_w)
\end{aligned}$$

$$\begin{aligned}
\underbrace{3 \times 3}_{SU(3)} &= S + A \\
&= 6 + 3^*
\end{aligned}$$

$$\epsilon_{ijk} 3_i 3_j 3_k = \text{singlet}$$

Proof:

$$\epsilon_{ijk} z_i z_j z_k \rightarrow \epsilon_{ijk} U_{i' i} U_{j' j} U_{k' k} \\ \times z_{i'} z_{j'} z_{k'}$$

but

$$\epsilon_{ij} U_{i' i} U_{j' j} U_{k' k} = \\ = \epsilon_{i' j' k'} \det U$$

$$\epsilon_{ijk} z_i z_j z_k \rightarrow \epsilon_{i' j' k'} z_{i'} z_{j'} z_{k'} \\ = \epsilon_{ijk} z_i z_j z_k$$

Q.E.D.

$$\Rightarrow (3 \times 3 \times 3)_{A5} \sim 1$$

$$(3 \times 3)_{A5} \sim 3^* \quad (3^* \cdot 3 \cdot 1)$$

$\Downarrow$

$$5 \times 5 = \underbrace{(6_c, 1_w)}_{15} + \underbrace{(3_c^*, 1_w)}_{10}$$

$$+ (3_c, 2_w) \quad (15, 10)$$

$$+ \underbrace{(1_c, 3_w)}_{15} + \underbrace{(1_c, 1_w)}_{10}$$

$$\Rightarrow 10_F = \underbrace{(3_c^*, 1_w)}_{u^c} + \underbrace{(3_c, 2_w)}_{u, d} + \underbrace{(1_c, 1_w)}_{e^c}$$

$$15_F = \underbrace{(6_c, 1_w)}_{?} + \underbrace{(3_c, 2_w)}_{(u, d)}$$

$$+ (1_c, 3_w)_{?}$$

NOT in nature



$S_M$  15 fermions =

$$= \bar{5}_F + 10_F$$

$\mathcal{S} \mathcal{S} B$  of  $SU(5)$

$$SU(5) \xrightarrow{M_{GUT}} G_{SM}$$

$r=4$        $\langle \Sigma \rangle$        $r=4$



new Higgs

$$G_{SM} \xrightarrow[\langle \phi \rangle]{-M_W} U(1)_Q \times SU(3)_c$$

$\swarrow$   
 SM doublet

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$\Sigma = \text{field} \therefore$

$\langle \Sigma \rangle$  keeps the rank



$\Sigma_{\text{minimal}} = \text{adjoint} \therefore$

$$\boxed{\Sigma \rightarrow U \Sigma U^\dagger} \quad (1)$$

$$(A \rightarrow U A U^T, S \rightarrow U S U^T)$$

•  $T_\nu \Sigma \rightarrow T_\nu \Sigma$  (inv.)

$\Rightarrow T_\nu \bar{\Sigma} = 0$  (minimal)

•  $\Sigma = \Sigma^\dagger$  is preserved under (1)



Adjoint:  $\Sigma \rightarrow U \Sigma U^\dagger$

$\Sigma = \Sigma^\dagger, T_\nu \Sigma = 0$

$\Rightarrow \Sigma = T_i \Psi_i$

$N^2 - 1$  in  $SU(N)$



$$\langle \Sigma \rangle = \Sigma_0 \therefore \Sigma_0 \rightarrow U \Sigma_0 U^\dagger$$

$$\Rightarrow \boxed{\langle \Sigma_0 \rangle = \text{diagonal}}$$

$$\Downarrow$$
$$[\langle \Sigma_0 \rangle, T_{\text{Cartan}}] = 0$$

$$\Leftrightarrow \Sigma_0 = (c_\alpha T_{\text{Cartan}}^\alpha) \text{ summed}$$

$$\Updownarrow$$
$$\boxed{\Sigma_0 = \langle \Sigma \rangle \text{ preserves the roots}}$$

$\Sigma_0 =$  preserves the SM sym.



$$\Sigma_0 = v_{\text{GUT}} \text{diag}(2, 2, 2, -3, -3)$$

Proof:

$$\Sigma \rightarrow U \Sigma U^\dagger$$

$$U = e^{i\theta T} = 1 + i\theta T + \dots$$

$$\Sigma \rightarrow \Sigma + i\theta_i [T_i, \Sigma]$$



$$\hat{T}_i \Sigma = [T_i, \Sigma]$$



$$\Rightarrow [T_c, \Sigma_0] = 0$$

$$[T_w, \Sigma_0] = 0$$

$$[Y, \Sigma_0] = 0$$

if  $\Sigma_0 = \text{diag.}$

$$\Rightarrow \left\{ \begin{array}{l} \Sigma_0 \propto [1_c, 1_w] \\ T_i \Sigma_0 = 0 \end{array} \right.$$



$$\underline{\Sigma_0 \propto [2, 2, 2, -3, -3]}$$

Q. E. D.

$$V_Z = -\frac{\mu^2}{2} T_V \Sigma^2 + \frac{a}{4} (T_V \Sigma^2)^2$$

$$+ \frac{b}{2} T_V \Sigma^4 \quad + \frac{\mu}{3} T_V \Sigma^3$$

$$(\Sigma \rightarrow -\Sigma)$$

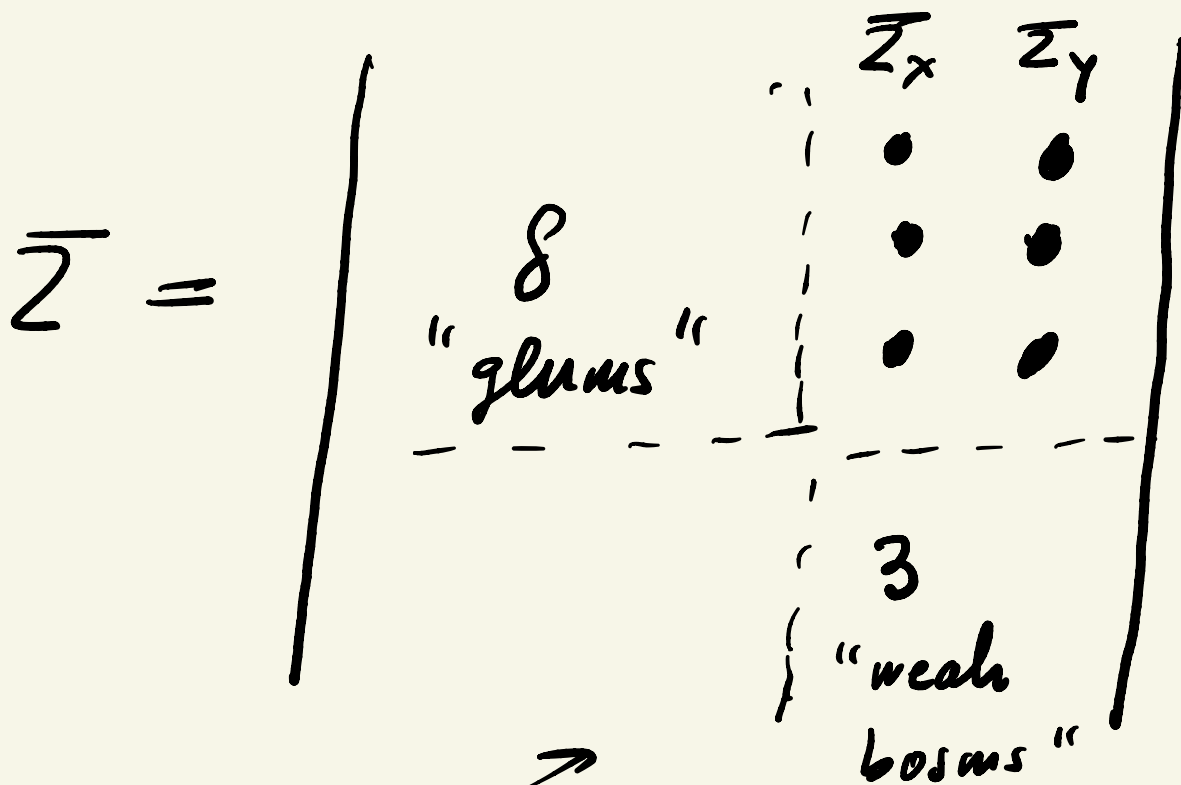
$\Downarrow$  ??

$$\mu^2 = (15a + 7b) v_{GUT}^2 \quad ??$$

extremum :  $15a + 7b > 0$

minimum?  $\Leftrightarrow (\text{mass})^2 > 0$ ?

$\Sigma_0 \leftarrow$  preserves  $SU(3)_c$ ,  
 $SU(2)_w$ ,  $U(1)$



$\rightarrow$   
+ singlet of all

"glue" =  $U_1$

$$\mu_{\bar{w}} = \mu_2$$

$$\mu_{\bar{s}} = \mu_3$$

$$\mu_{\Sigma x} = \mu_{\Sigma y} = \mu_4$$