GUT Course 22/23

Lecture XIII

6/12/2022

LMU

Fall 2022

SU(5) GUT (2)

$$G_{SM} = SU(3)_c \times SU(2)_L \times U(1)_Y$$

$$\boxed{G_{SM} \subseteq G_{GUT} = \underline{\text{grand}} \text{ unification}}$$

- $G_{GUT}^{min} = SU(5)$
- $r(G_{SM}) = r(SU(5)) = 4$
- $G_{SM} = H_{max} \quad (H \subseteq SU(5))$

NO larger $H \supseteq H_{max}$

- another $H_{max} = SU(4) \times U(1)$

- Cartan $= \{ \underbrace{T_{3c}, T_{8c}}_{SU(3)_c}, \underbrace{T_{3w}}_{SU(2)_L}, \underbrace{Y}_{U(1)} \}$

$$T_{3c} = \mathrm{diag}\, \frac{1}{2} (\underbrace{1, -1, 0}_{\text{color}} ; 0, 0)$$

$$T_{8c} = \mathrm{diag}\, \frac{1}{2\sqrt{3}} (1, 1, -2 ; 0, 0)$$

$$T_{3w} = \mathrm{diag}\, \frac{1}{2} (0, 0, 0 ; 1, -1)$$

$$Y \overset{\text{norm}}{=} \mathrm{diag}\, N (1, 1, 1 ; -3/2, -3/2)$$

$SU(5)$

$$\underline{5} \longrightarrow U\,\underline{5} \quad \therefore$$

$$UU^\dagger = U^\dagger U = 1 \quad , \det U = 1$$

$$\Rightarrow \quad U = e^{iH} = e^{i\theta_i T_i}$$

$H^\dagger = H$ $\qquad$ $i = 1, \ldots, 24$

$\mathrm{Tr}\, H = 0$ $\qquad$ $T_i^\dagger = T_i, \ \mathrm{Tr}\, T_i = 0$

$$\boxed{\mathrm{Tr}\, T_i T_j = \frac{1}{2}\delta_{ij}}$$

$$[T_i, T_j] = i f_{ijk} T_k$$

$$24 = 20 + 4$$

$\sigma_{1,2}$ all over $\qquad$ Cartan

- $T_i \longleftrightarrow A_i$

$$D_\mu 5 = (\partial_\mu - ig\, T_i A_\mu^i) 5$$

$$24 = 12 + 12$$

new

8 gluons, W^+, W^-, Z, A = SM

- Matter = fermions

$\alpha = 1,2,3$
$= r, y, b$

$$\begin{pmatrix} u \\ d \end{pmatrix}_L^\alpha \equiv q \, ; \quad (u^c)_L^\alpha, \ (d^c)_L^\alpha$$

$$\ell \equiv \begin{pmatrix} \nu \\ e \end{pmatrix}_L \, ; \quad (e^c)_L$$

$$\bar{5}_F = \begin{pmatrix} d^c \\ d^c \\ d^c \\ \hline \ell \end{pmatrix}_L \begin{matrix} \left.\vphantom{\begin{matrix} d^c \\ d^c \\ d^c \end{matrix}}\right\} SU(3)_c \\ \left.\vphantom{\ell}\right\} SU(2)_W \end{matrix}$$

$$10_F = \begin{pmatrix} 0 & u^c & u^c & u & d \\ & 0 & u^c & u & d \\ & & 0 & u & d \\ & & & 0 & e^c \\ & & & & 0 \end{pmatrix}_L$$

As

$$\boxed{5 \times 5 = \underbrace{15}_{(S)\ \text{sym}} + \underbrace{10}_{\text{antisym (AS)}}}$$

$$[5_i 5_j] = A_{ij}$$

$$\Rightarrow \boxed{A \to U A U^T}$$

$$\Leftrightarrow \quad A_{ij} \to U_{ik} A_{kl} U^T_{lj}$$

$$= U_{ik} U_{jl} A_{kl}$$

$$\Downarrow$$

$$\boxed{\hat{T} A = T A + U T^T}$$

$$\Downarrow$$

$$\hat{Q} A = Q A + A Q$$

$$\Rightarrow \boxed{(\hat{Q} A)_{ij} = (q_i + q_j) A_{ij}}$$

$$\bar{5}_F = \begin{pmatrix} d^c \\ --- \\ e \end{pmatrix}_L \qquad (\bar{5}_F = 5_F{}^c)$$

$$\Leftrightarrow \quad 5_F = \begin{pmatrix} d \\ --- \\ e^c \\ \nu^c \end{pmatrix}_R \qquad \begin{aligned} (e^c)_R &\equiv C \bar{e}_L{}^T \\ (\nu^c)_R &\equiv C \bar{\nu}_L{}^T \end{aligned}$$

$$Q \equiv Q_{em}$$

$$\Downarrow$$

$$Q = \mathrm{diag}\left(-1/3, -1/3, -1/3; 1, 0\right)$$

$$T_{3w} = \mathrm{diag}\left(0, 0, 0; 1, -1\right)$$

$$\frac{Y}{2} = \mathrm{diag}\left(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}; \frac{1}{2}, \frac{1}{2}\right)$$

$\boxed{\text{Charge quantized}}$

5_F :

$$3\,Q(d) - Q(e) - Q(\nu) = 0$$

$$\Uparrow$$

$$\mathrm{Tr}\, Q = 0$$

$$\Rightarrow \quad Q(e) + Q(\nu) = 3\,Q(d)$$

$$Q(\nu) = Q(e) + 1$$

$$\boxed{\begin{aligned} &Q = T_3 + Y/2 \\ &\Rightarrow \quad Q(u) - Q(d) = 1 \end{aligned}}$$

$5_F \Rightarrow \boxed{2\,Q(e) + 1 = 3\,Q(d)}$

$T_1 Q = 0$

$$10_F \Rightarrow \quad Q(u^c) = 2\,Q(d)$$

$$\parallel$$

$$-Q(u) = -\left(Q(d) + 1\right)$$

$$\Rightarrow \boxed{3\,Q(d) = -1}$$

$$\Rightarrow \boxed{\begin{aligned} Q(d) &= -1/3 \\ Q(e) &= -1 \\ Q(\nu) &= 0 \\ Q(u) &= 2/3 \end{aligned}}$$

$$\Leftarrow \quad \mathrm{Tr}\, Q(10_F) = 0$$

$$\Downarrow$$

$$3\left(Q(u^c) \cancel{+ Q(u)} + Q(d)\right) + Q(e^c) = 0$$

$$\boxed{3\,Q(d) = -Q(e^c) = Q(e)}$$

$$\boxed{\text{Charge is quantized in } SU(5)}$$

$\Rightarrow$ magnetic monopole!

$$\boxed{\text{weak mixing angle } (\theta_W)}$$

SM: $D_\mu = \cdots - i g\, T_i A_\mu^i - i g' \frac{Y}{2} B$

$$= \cdots - i \left(g\, T_3 A_3 + g' \frac{Y}{2} B \right)_\mu$$

$$\tan \theta_W = g'/g$$

$$\mathrm{Tr}\, T_3^2 = \frac{1}{2}$$

normalized gen.

SU(5): $g' \frac{Y}{2} = g_1 I_1$

$$\underbrace{g_1 = g_2 = g_3}_{\text{unification}} \equiv g \equiv g_5$$

$$\equiv g_{QCD}$$

iff: $\boxed{T_1 I_1^2 = \frac{1}{2}}$

but:

$$T_r\left(\frac{Y}{2}\right)^2 = \frac{1}{9}\cdot 3 + \frac{1}{4}\cdot 2 = \frac{1}{3} + \frac{1}{2}$$

$$= 5/6 = \frac{5}{3}\ 1/2$$

$$\boxed{\frac{Y}{2} = \sqrt{5/3}\ I_1}$$

$$\therefore \quad T_r\left(\frac{Y}{2}\right)^2 = \frac{5}{3}\, T_r I_1^2 =$$

$$= 5/3 \cdot 1/2 = 5/6$$

but: $g_1 I_1 = g' \frac{Y}{2}$

$$\Rightarrow \quad g' = \sqrt{3/5}\, g_1 = \sqrt{3/5}\, g$$

$$g_2 = g$$

$$\Downarrow$$

$$\tan^2\theta_W = \left(\frac{g'}{g_2}\right)^2 = \frac{3}{5}$$

$$\Downarrow$$

$$\boxed{\sin^2\theta_W = \frac{3}{8}, \qquad \cos^2\theta_W = \frac{5}{8}}$$

WRONG??!!

$\boxed{\text{NO}}$

- $g_2 = g_3 = g_1 = g$

at M_{GUT}

unification scale

$$\Downarrow$$

$$\sin^2\theta_W\,(M_{GUT}) = 3/8$$

$$\Downarrow$$

$$\sin^2\theta_W\,(M_W) = 3/8 \pm \cdots$$

"running"

$$\alpha = \alpha(E)$$

$$\frac{1}{\alpha(E_2)} = \frac{1}{\alpha(E_1)} + \frac{b}{2\pi}\ln E_2/E_1$$

"crawling" :)

$$(\underline{5} \times \underline{5}) = (\underline{5} \times \underline{5})_S + (\underline{5} \times \underline{5})_{AS}$$

$$= 15 + 10$$

$$\boxed{\text{all SM fermions in } 15_F\,?}$$

<u>Inspection</u>:

$$5 = \underbrace{\overset{①}{(3_c, 1_W)} + \overset{②}{(1_c, 2_W)}}$$

$$\Rightarrow 5 \times 5 = \sim \times \sim$$

$$= 1 \times 1 + 2 \times 2 +$$

$$+ 1 \times 2 + 2 \times 1$$

$$= (3_c \times 3_c, 1_w) + (1_c, 2_w \times 2_w)$$

$$+ (3_c, 2_w)$$

$$\underbrace{3 \times 3}_{SU(3)} = S + A$$

$$= 6 + 3^*$$

$$\varepsilon_{ijk}\, 3_i\, 3_j\, 3_k = \text{singlet}$$

<u>**Proof:**</u>

$$\varepsilon_{ijk}\, \zeta_i \zeta_j \zeta_k \to \varepsilon_{ijk}\, U_{ii'} U_{jj'} U_{kk'}$$

$$\times\, \zeta_{i'} \zeta_{j'} \zeta_{k'}$$

<u>but</u>

$$\boxed{\begin{aligned} &\varepsilon_{ijk}\, U_{ii'} U_{jj'} U_{kk'} = \\ &\quad = \varepsilon_{i'j'k'} \det U \end{aligned}}$$

$$\varepsilon_{ijk}\, \zeta_i \zeta_j \zeta_k \to \varepsilon_{i'j'k'}\, \zeta_{i'} \zeta_{j'} \zeta_{k'}$$

$$= \varepsilon_{ijk}\, \zeta_i \zeta_j \zeta_k$$

Q.E.D.

$$\Rightarrow \quad (3 \times 3 \times 3)_{AS} \sim 1$$

$$(3 \times 3)_{AS} \sim 3^* \qquad (3^* \cdot 3 \sim 1)$$

$$\Downarrow$$

$$5 \times 5 = \underbrace{(6_c, 1_w)}_{15} + \underbrace{(3^*_c, 1_w)}_{10}$$

$$+ (3_c, 2_w) \qquad (15, 10)$$

$$+ \underbrace{(1_c, 3_w)}_{15} + \underbrace{(1_c, 1_w)}_{10}$$

$$\Rightarrow \quad 10_F = \underbrace{(3^*_c, 1_w)}_{u^c} + \underbrace{(3_c, 2_w)}_{u,\, d} + \underbrace{(1_c, 1_w)}_{e^c}$$

$$15_F = \underbrace{(6_c, 1_w)}_{?} + \underbrace{(3_c, 2_w)}_{(u,\, d)} + \underbrace{(1_c, 3_w)}_{?}$$

NOT in nature

$\Downarrow$

$$\boxed{\text{SM } 15 \text{ fermions} = \\ = \bar{5}_F + 10_F}$$

SSB of SU(5)

$$\underset{r=4}{SU(5)} \xrightarrow[\langle \Sigma \rangle]{M_{GUT}} \underset{r=4}{G_{SM}}$$

new Higgs

$$G_{SM} \xrightarrow[\langle \phi \rangle]{M_W} U(1)_Q \times SU(3)_c$$

SM doublet

Σ = field :.

$\langle \Sigma \rangle$ keeps the rank

$\Downarrow$

Σ minimal = adjoint :.

$$\boxed{\Sigma \to U \Sigma U^{+}} \qquad (1)$$

$(A \to U A U^T, \; S \to U S U^T)$

- $\mathrm{Tr}\,\Sigma \to \mathrm{Tr}\,\Sigma$ (inv.)

$$\Rightarrow \quad \mathrm{Tr}\,\Sigma = 0 \text{ (minimal)}$$

- $\Sigma = \Sigma^{+}$ is preserved under (1)

$$\Downarrow$$

Adjoint: $\Sigma \to U \Sigma U^{+}$

$$\Sigma = \Sigma^{+}, \; \mathrm{Tr}\,\Sigma = 0$$

$$\Rightarrow \quad \Sigma = T_i \varphi_i$$

$N^2 - 1$ in $SU(N)$

$$\langle \Sigma \rangle = \Sigma_0 \therefore \Sigma_0 \to U \Sigma_0 U^{+}$$

$$\Rightarrow \boxed{\langle \Sigma_0 \rangle = \text{diagonal}}$$

$$\Downarrow$$

$$[\langle \Sigma_0 \rangle , T_{\text{cartan}}] = 0$$

$$\Leftarrow \quad \Sigma_0 = (c_\alpha T^\alpha_{\text{cartan}})\ \text{summed}$$

$$\Updownarrow$$

$\Sigma_0 = \langle \Sigma \rangle$ preserves

the rank

Σ_0 = preserves the SM symm.

$$\Downarrow$$

$$\boxed{\Sigma_0 = v_{GUT}\, \mathrm{diag}(2,2,2,-3,-3)}$$

Proof:

$$\Sigma \to U \Sigma U^+$$

$$U = e^{i\theta T} = 1 + i\theta T + \cdots$$

$$\Sigma \to \Sigma + i\theta_i [T_i, \Sigma]$$

$$\Updownarrow$$

$$\hat{T}_i \Sigma = [T_i, \Sigma]$$

$$\Downarrow$$

$$\Rightarrow [T_c, \Sigma_0] = 0$$

$$[T_w, \Sigma_0] = 0$$

$$[Y, \Sigma_0] = 0$$

if $\Sigma_0 = \text{diag.}$

$$\Rightarrow \begin{cases} \Sigma_0 \propto [\mathbb{1}_c, \mathbb{1}_w] \\ T_1 \Sigma_0 = 0 \end{cases}$$

$$\Downarrow$$

$$\boxed{\Sigma_0 \propto [2, 2, 2, -3, -3]}$$

Q. E. D.

$$V_\Sigma = -\frac{\mu^2}{2} Tr\,\Sigma^2 + \frac{a}{4}(Tr\,\Sigma^2)^2$$

$$+ \frac{b}{2} Tr\,\Sigma^4 \quad \cancel{+ \frac{\mu}{3} Tr\,\Sigma^3}$$

$$(\Sigma \to -\Sigma)$$

$\Downarrow$??

$$\mu^2 = (15a + 7b)\, v_{GUT}^2 \quad ??$$

extremum : $15a + 7b > 0$

$\underline{\text{minimum}}$? $\Leftrightarrow$ $(\text{mass})^2 > 0$?

$\Sigma_0 \leftarrow$ preserves $SU(3)_c$, $SU(2)_w$, $U(1)$

$$\Sigma = \begin{array}{c} \begin{array}{cc} \phantom{8 \text{ "gluons"}} & \Sigma_X \quad \Sigma_Y \end{array} \\ \left|\begin{array}{c|c} \begin{matrix} 8 \\ \text{"gluons"} \end{matrix} & \begin{matrix}\bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet\end{matrix} \\ \hline & \begin{matrix} 3 \\ \text{"weak} \\ \text{bosons"} \end{matrix} \end{array}\right| \end{array}$$

+ singlet at all

$m_{\text{"glue"}} = m_s$

$$\mathfrak{m}_{\text{"}\bar{w}\text{"}} = \mathfrak{m}_2$$

$$\mathfrak{m}_{\text{"}s\text{"}} = \mathfrak{m}_3$$

$$\mathfrak{m}_{\Sigma x} = \mathfrak{m}_{\Sigma y} = \mathfrak{m}_4$$