GUT Course 22/23

Lecture VIII

18/11/2022

LHU

Fall 2022

Magnetic Monopoles

$$G_{GUT} \xrightarrow[M_{GUT}]{} SU(3) \times SU(2) \times \underline{U(1)}$$

$$\downarrow M_W$$

$$SU(3) \times \underline{U(1)}$$

$$(SU(2) \cong) \boxed{SO(3) \longrightarrow U(1)}$$

$$\boxed{\text{vector Higgs} = \text{Triplet}}$$

$$\phi \to O\,\phi$$

$$O O^T = O^T O = \mathbb{1}$$

$$\det O = 1$$

$$O = e^{i\theta_i L_i}$$

$$O^T O = 1 \Rightarrow (L_i)_{jk} = -(L_i)_{kj}$$

$$(L_i)_{jk} = -i\,\varepsilon_{ijk}$$

$$\Updownarrow$$

$$[L_i, L_j] = i\,\varepsilon_{ijk} L_k$$

<u>digression</u> $SU(2)$: $\Sigma \to U \Sigma U^\dagger$

$$\mathrm{tr}\,\Sigma = 0, \quad \Sigma = \Sigma^\dagger$$

$$\Sigma = \varphi_i T_i \qquad (T_i = \sigma_i/2)$$

$$\phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = \text{vector}$$

$$\mathcal{L} = \frac{1}{2}(D_\mu \phi)^T (D^\mu \phi) - V(\phi)$$

$$- \frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} \qquad a = 1,2,3$$

$$D_\mu \phi = \partial_\mu - ig\, L_a A_\mu^a$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\, \epsilon_{abc} A_\mu^b A_\nu^c$$

$$V = \frac{\lambda}{4}\left(\phi^T \phi - v^2\right)^2$$

$$\mathcal{M}_0 = \{ \phi_0 \therefore \phi_0^T \phi_0 = v^2 \} =$$

$$= \{ \phi_0^a \therefore \phi_0^a \phi_0^a = v^2 \}$$

$$= S_2$$

$$SO(3) \xrightarrow[\phi_0]{} SO(2) = U(1)$$

$$(L_a)_{bc} = -i\,\varepsilon_{abc}$$

$$D_\mu \phi_0 \to -ig\,(L_a)\,A_\mu^a\,\phi_0$$

$$\frac{1}{2}(D_\mu \phi_0)^T (D^\mu \phi_0) = \frac{1}{2}(D_\mu \phi_0)^b (D^\mu \phi_0)_b$$

$$(D_\mu \phi_0)_b = -ig\,(L_a)_{bc}\,A_\mu^a\,\phi_0^c$$

$$= -g\,\varepsilon_{abc}\,A_\mu^a\,\phi_0^c$$

$$\Downarrow$$

$$\frac{1}{2} g^2 \varepsilon_{amc} A_\mu^a \phi_0^c\, \varepsilon_{bmd} A^\mu_b \phi_0^d$$

$$= \frac{1}{2} g^2 (\delta_{ab}\delta_{cd} - \delta_{ad}\delta_{bc}) A_\mu^a A^\mu_b \phi_0^c \phi_0^d$$

$$\boxed{\underline{M}^2_{ab}(A) = (\delta_{ab}\,\phi_0^2 - \phi_a^0\,\phi_b^0)\,g^2}$$

$$A_\mu = A_\mu^a \,\frac{\phi_0^a}{|\phi_0|} \Longleftrightarrow$$

$$\underline{M}^2_{ab}\;\phi_0^b = 0$$

$=$ photon (massless gauge boson)

$\cdot\ \phi_0^b = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} \Rightarrow \quad A_\mu = A_\mu^3\,,$

$\Leftarrow \Big/ L_3\,\phi_0^b = 0 \Leftrightarrow M_{A_3} = 0$

$$\Rightarrow \boxed{M_{A_1} = M_{A_2} = g v} \boxed{\Leftarrow SO(2)}$$

$$D_\mu \phi_0 \Rightarrow g \begin{pmatrix} -A_2 \\ A_1 \\ 0 \end{pmatrix} v$$

L_3 = unbroken $\Leftrightarrow$ rotation around L_3

- $A_\mu \equiv A_\mu{}^3$ (photon)

$$\begin{aligned} F_{\mu\nu} \neq (F_{\mu\nu})^3 &= \partial_\mu A_\nu{}^3 - \partial_\nu A_\mu{}^3 + g\left(A_\mu{}^1 A_\nu{}^2 - 1 \to 2\right) \\ \| \\ \partial_\mu A_\nu{}^3 - \partial_\nu A_\mu{}^3 &= \partial_\mu A_\nu - \partial_\nu A_\mu \end{aligned}$$

$$\boxed{\begin{array}{c} \text{Q. } \phi_0{}^a \rightarrow \text{what is } F_{\mu\nu}? \\ \Uparrow \\ A_\mu = A_\mu{}^a \dfrac{\phi_0{}^a}{|\phi_0|} \end{array}}$$

A. $F_{\mu\nu}$ = covariant form

$\therefore \; F_{\mu\nu} = \partial_\mu A_\nu{}^3 - \partial_\nu A_\mu{}^3$ for $\phi_0^a = v\,\delta_{a3}$

$$\Downarrow$$

$$F_{\mu\nu} = \frac{F_{\mu\nu}{}^a \, \phi_0{}^a}{|\phi_0|} - \frac{\varepsilon_{abc} \, \phi_0{}^a (D_\mu\phi_0)^b (D_\nu\phi_0)^c}{g \, |\phi_0|^3}$$

* Prove it !

$\underline{a=3}$

$$\Uparrow$$

$$F_{\mu\nu} = F_{\mu\nu}^{3} - \frac{(D_\mu \phi_0)_1 (D_\nu \phi_0)_2}{g\, |\phi_0|^2}$$

$$\|$$

$$\partial_\mu A_\nu^3 - \partial_\nu A_\mu^3 + g\left(A_\mu^1 A_\nu^2 - A_\nu^1 A_\mu^2\right)$$

$$Q_{em} = L_3 \quad \therefore \quad \phi_0^{G} = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}$$

general form:

$$\boxed{Q_{em} = \frac{L_a \phi_0^a}{|\phi_0|}}$$

$$(Q_{em}\,\phi_0)_a = \frac{(L_b\,\phi_0^b)_{ac}\,\phi_{0c}}{|\phi_0|}$$

$$\shortparallel$$

$$(Q_{em})_{ac}\,\phi_0^c$$

$$= -\frac{i\,\varepsilon_{bac}\,\phi_0^b\,\phi_0^c}{|\phi_0|} = 0$$

Q.E.D.

$$\boxed{\text{Static solution, finite } E}$$

$$E = \int d\bar{V} \left[\frac{1}{2}(D_i\phi)^2 + V(\phi) + \frac{1}{2}(\vec{E}_a^{\,2} + \vec{B}_a^{\,2}) \right]$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

at ∞ $\quad 0 \quad 0 \quad 0$

$$\phi_\infty : \quad V(\phi_\infty) = 0$$

$$D_i(\phi_\infty) = 0$$

$$\begin{tikzcd}
\mathcal{M}_\infty \arrow[r] \arrow[d, phantom, "\shortparallel"] & \mathcal{M}_0 \arrow[d, phantom, "\shortparallel"] \\
S_2 & S_2
\end{tikzcd}$$

well-defined map

• $\phi_\infty^a = \phi_0^a$ vacuum

$$\Downarrow$$

$$\phi^a = \phi_0^a \text{ everywhere} \qquad (n=0)$$

- $$\phi_\infty^a = v\,\frac{x^a}{r} \Rightarrow \phi^a(0) = 0$$

$\phi_0 = 0$

$r \ll \delta$

$$\underbrace{\phi_\infty^a = v\,\frac{x_a}{r} \Rightarrow \phi_0^a}_{r \gg \delta}$$

$$\phi_\infty^z = v\cos\theta \quad = \quad \phi_\infty^z$$

$$\phi_\infty^1 = v\sin\theta\cos\phi = \phi_\infty^x$$

$$\phi_\infty^2 = v\sin\theta\sin\phi$$

$(n = 1 \text{ map})$

n times around M_0

$$\phi_\infty^3 = v\cos\theta$$

$$\phi_\infty^1 = v\sin\theta\cos n\phi$$

$$\phi_\infty^2 = v\sin\theta\sin n\phi$$

covering the sphere n times

$\boxed{n=1}$ = magnetic monopole

Proof:

$$Q_{em}^\infty = L_a\, {}^{x_a}/_v$$

$$A_\mu^\infty = A_\mu^a \, x_a / r$$

$$F_{\mu\nu}^\infty = F_{\mu\nu}^a(\infty) \frac{x_a}{r} - \frac{1}{g} \epsilon_{abc} \frac{\phi_\infty^a (D_\mu \phi)_b^\infty (D_\nu \phi)_c^\infty}{|\phi_\infty|^3}$$

$O(1/r^2)$

neglect $\left(\xrightarrow[1/r^4]{} 0 \right)$

$\Downarrow$

$$\boxed{F_{\mu\nu}^\infty \longrightarrow F_{\mu\nu}^a(\infty) \frac{x_a}{r}}$$

Compute it!

$$(D_i \phi)^\infty = 0$$

$$\Rightarrow \quad \partial_i \phi^\infty - ig(L_a A_a^i)\phi^\infty = 0$$

$$\phi_b^\infty = v \, x_b/r$$

$$\partial_i \left(x_b/r\right) = ig(L_a)_{bc} A_i^a \cdot x_c/r$$

$$= -g \, \epsilon_{abc} A_i^a \, x_c/r$$

$$A_i^a = a \frac{\delta_i^a}{r} + b \, \varepsilon_{iab} \, x_b/r$$

$$\partial_i \left(x_b/r\right) = \frac{\delta_{ib}}{r} - \frac{x_i x_b}{r^3}$$

$$\boxed{A_i^a = \frac{1}{g}\,\varepsilon_{iab}\, x_b/r^2 \quad (\pm)} \quad \boxed{\text{Prove!}}$$

$$\partial_i\, x_b/r = -g\, \epsilon_{abc}\, A_i^a\, x_c/r\, \frac{1}{g}$$

$$= -g\, \epsilon_{abc}\, \varepsilon_{iad}\, \frac{x_c\, x_d}{r^2}\, \frac{1}{g}$$

$$= +g\, \varepsilon_{abc}\, \varepsilon_{aid}\, \frac{x_c\, x_d}{r^2}\, \frac{1}{g}$$

$$= g\, (\delta_{ib}\delta_{cd} - \delta_{ci}\delta_{bd})\, \frac{x_c\, x_d}{r^2}\, \frac{1}{g}$$

$$= \cancel{g}\, \frac{\delta_{ib}\, r^2 - x_i\, x_b}{r^2}\, \cancel{\frac{1}{g}} \quad \checkmark$$

$$\Downarrow$$

$$F^{\infty}_{ij} = F^{\infty\, a}_{ij}\ x_a/r =$$

$$= \left(\partial_i A^{\infty\, a}_j - \partial_j A^{\infty\, a}_i + g\, \varepsilon_{abc} A^{\infty\, b}_i A^{\infty\, c}_j\right)\frac{x_a}{r}$$

$$= \frac{1}{g}\left(\partial_i\left(\varepsilon_{jab}\frac{x_b}{r^2}\right) - \partial_j\left(\varepsilon_{iab}\frac{x_b}{r^2}\right)\right.$$

$$\left. + \varepsilon_{abc}\,\varepsilon_{ibk}\ x_k/r^2\ \varepsilon_{jce}\frac{x_e}{r^2}\right)\frac{x_a}{r}$$

$$= \frac{1}{g}\left(\underline{\varepsilon_{jab}}\left(\frac{\delta_{ib}}{r^2} + \cancel{\sim\frac{x_i x_b}{\dots}}\right)\right)\frac{x_a}{r}$$

$$- \underline{\varepsilon_{iab}\ \delta_{jb}/r^2\ x_a/r}$$

$$+ \left(\delta_{ai}\delta_{ck} - \delta_{ak}\delta_{ic}\right)\varepsilon_{jce}\frac{x_k x_e x_a}{r^5}\Big)$$

$$= \frac{1}{g}\left(-2\epsilon_{ija}\frac{x_a}{r^3} - \varepsilon_{jil}\frac{x_l r^2}{r^5}\right)$$

$$= \frac{1}{g}\left(-2\epsilon_{ija}\frac{x_a}{r^3} + \varepsilon_{ijl}\frac{x_l}{r^3}\right)$$

$$\Downarrow$$

$$F_{ij}^{8} = -\frac{1}{g}\,\epsilon_{ija}\frac{x_a}{r^3}$$

$$\parallel$$

$$\varepsilon_{ija} B_a^{8} \Rightarrow \boxed{\vec{B}_a^{\,8} = \frac{1}{g}\frac{\hat{r}}{r^2}}$$

$$\Downarrow$$

magnetic monopole

$$\vec{B}(g_m) = \frac{g_m}{4\pi r^2}\hat{r}$$

$$\boxed{g\, g_m \doteq 4\pi}$$

$\boxed{\text{Q. E. D.}}$

1948 (?)
Dirac

$\exists$ magnetic monopole

$\Downarrow$

QM of e in field of

$\Downarrow$

$$\boxed{q_e\, g_m = 2\pi n}$$

$n = 1, 2, \ldots$

$SO(3) \Rightarrow Q_{em} =$ quantized

$\downarrow$

$SU(2) \leftarrow$ spinor