

GUT Course 22/23

Lecture VIII

18/11/2022

L M U

Fall 2022



Magnetic Monopoles

$$\begin{array}{ccc} \underline{G_{GUT}} & \longrightarrow & SU(3) \times SU(2) \times \underline{U(1)} \\ & \text{M}_{GUT} & \downarrow \text{M}_W \\ & & SU(3) \times \underline{U(1)} \end{array}$$

$$(SU(2) \cong) \left[\begin{array}{ccc} SO(3) & \longrightarrow & U(1) \\ \text{vector Higgs} & = & \text{Triplet} \end{array} \right]$$

$$\phi \rightarrow O \phi$$

$$O O^T = O^T O = \mathbb{1}$$

$$\det O = 1$$

$$O = e^{i\theta_i L_i}$$

$$O^T O = 1 \Rightarrow (L_i)_{ju} = - (L_i)_{uj}$$

$$(L_i)_{ij} = -i \epsilon_{ijk}$$

$$[L_i, L_j] = i \epsilon_{ijk} L_k$$

diagonalization

$$SU(2) \therefore \Sigma \rightarrow U \Sigma U^\dagger$$

$$\text{tr } \Sigma = 0, \quad \Sigma = \Sigma^\dagger$$

$$\Sigma = \varphi_i T_i \quad (T_i = \sigma_i/2)$$

$$\phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = \text{vector}$$

$$\mathcal{L} = \frac{1}{2} (D_\mu \phi)^T (D^\mu \phi) - V(\phi) - \frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} \quad a=1,2,3$$

$$D_\mu \phi = \partial_\mu \phi - ig L_a A_\mu^a$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g \epsilon_{abc} A_\mu^b A_\nu^c$$

$$V = \frac{\lambda}{4} (\phi^T \phi - v^2)^2$$

$$M_0 = \left\{ \phi_0 \because \phi_0^T \phi_0 = v^2 \right\} =$$

$$= \left\{ \phi_0^a \because \phi_0^a \phi_0^a = v^2 \right\}$$

$$= S_2$$

$$SO(3) \xrightarrow{\phi_0} SO(2) = U(1)$$

$$(L_a)_{bc} = -i \Sigma_{abc}$$

$$D_\mu \phi_0 \rightarrow -ig (L_a) A_\mu^a \phi_0$$

$$\frac{1}{2} (D_\mu \phi_0)^T (D^\mu \phi_0) = \frac{1}{2} (D_\mu \phi_0)_b (D^\mu \phi_0)_b$$

$$(D_\mu \phi_0)_b = -ig (L_a)_{bc} A_\mu^a \phi_0^c$$

$$= -g \Sigma_{abc} A_\mu^a \phi_0^c$$

↓

$$\frac{1}{2} g^2 \varepsilon_{abc} A_\mu^a \phi_0^c \varepsilon_{bcd} A^\mu_b \phi_0^d$$

$$= \frac{1}{2} g^2 (\delta_{ab} \delta_{cd} - \delta_{ad} \delta_{bc}) A_\mu^a A^\mu_b \phi_0^c \phi_0^d$$

$$M_{ab}^2(A) = (\delta_{ab} \phi_0^2 - \phi_a^0 \phi_b^0) g^2$$

$$A_\mu = A_\mu^a \frac{\phi_0^a}{|\phi_0|} \Leftrightarrow$$

$$M_{ab}^2 \phi_0^b = 0$$

= photon (massless gauge boson)

$$\bullet \phi_0^6 = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} \Rightarrow A_\mu = A_\mu^3,$$

$$\Leftrightarrow \int L_3 \phi_0^6 = 0 \Leftrightarrow M_{A_3} = 0$$

$$\Rightarrow \boxed{M_{A_1} = M_{A_2} = g v} \quad \Leftrightarrow \boxed{SO(2)}$$

$$D_\mu \phi_0 \Rightarrow g \begin{pmatrix} -A_2 \\ A_1 \\ 0 \end{pmatrix} v$$

$L_3 = \text{unbroken} \Leftrightarrow \text{rotation around } L_3$

• $A_\mu = A_\mu^3 \quad (\text{photon})$

$$F_{\mu\nu} \neq (F_{\mu\nu})^3 = \partial_\mu A_\nu^3 - \partial_\nu A_\mu^3 + g (A_\mu^1 A_\nu^2 - 1 \rightarrow 2)$$

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$$\partial_\mu A_\nu^3 - \partial_\nu A_\mu^3 = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Q. $\phi_0^a \rightarrow$ what is $F_{\mu\nu}$?

$$A_\mu = \hat{\pi}^a \frac{\phi_0^a}{|\phi_0|}$$

A. $F_{\mu\nu} =$ covariant form

$$\therefore F_{\mu\nu} = \partial_\mu A_\nu^3 - \partial_\nu A_\mu^3 \text{ for } \phi_0^a = v \delta_{a3}$$



$$F_{\mu\nu} = \frac{F_{\mu\nu}^a \phi_0^a}{|\phi_0|} - \frac{\epsilon_{abc} \phi_0^a (D_\mu \phi_0)^b (D_\nu \phi_0)^c}{|\phi_0|^3}$$

* Prove it!

$$\underline{a=3}$$



$$F_{\mu\nu} = F_{\mu\nu}^3 - \frac{(D_\mu \phi_0)_1 (D_\nu \phi_0)_2}{g |\phi_0|^2}$$

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$$\partial_\mu A_\nu^3 - \partial_\nu A_\mu^3$$

$$+ g (A_\mu^1 A_\nu^2 - A_\nu^1 A_\mu^2)$$

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$$Q_{em} = L_3 \quad \therefore \quad \phi_0^G = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}$$

general form:

$$Q_{em} = \frac{L_a \phi_0^a}{|\phi_0|}$$

$$\begin{aligned}
 (Q_{em} \phi_0)_a &= \frac{(L_b \phi_0^b)_{ac} \phi_0^c}{|\phi_0|} \\
 \parallel \\
 (Q_{em})_{ac} \phi_0^c &\xrightarrow{\quad} \\
 &= \frac{-i \epsilon_{bac} \phi_0^b \phi_0^c}{|\phi_0|} = 0
 \end{aligned}$$

Q.E.D.

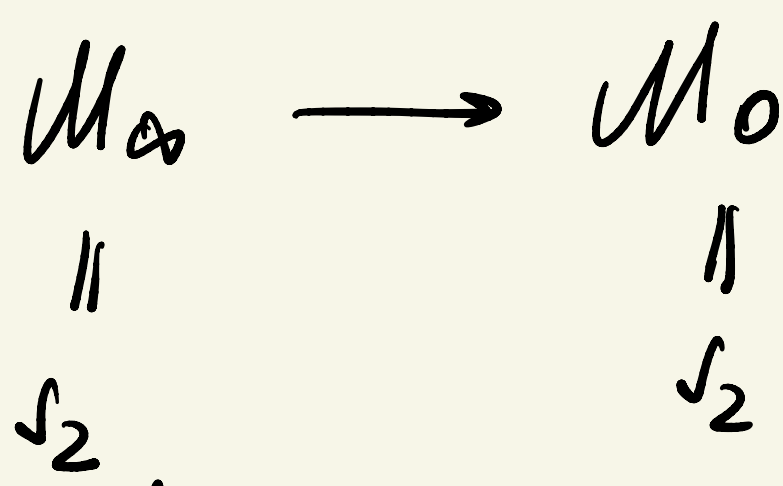
Static solution, finite E

$$E = \int dV \left[\underbrace{\frac{1}{2} (D_i \phi)^2}_{\downarrow} + \underbrace{V(\phi)}_{\downarrow} + \frac{1}{2} (\underbrace{\vec{E}_a^2 + \vec{B}_a^2}_{\downarrow}) \right]$$

at ∞ 0 0 0

$$\phi_\infty \therefore V(\phi_\infty) = 0$$

$$D_i(\phi_\infty) = 0$$



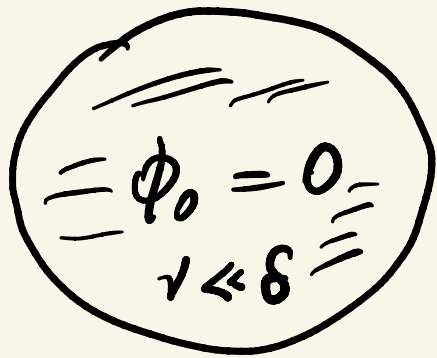
well-defined map

• $\phi_\infty^a = \phi_0^a$ vacuum



$$\phi^a = \phi_0^a \text{ everywhere } (u=0)$$

$$\phi_\infty^a = v \frac{x^a}{r} \Rightarrow \phi^a(0) = 0$$



$$\phi_\infty^a = v \frac{x^a}{r} \Rightarrow \phi_0^a$$

$\underbrace{\hspace{10em}}$
 $r \gg \delta$

$$\phi_\infty^z = v \cos \theta = \phi_\infty^z$$

$$\phi_\infty^1 = v \sin \theta \cos \phi = \phi_\infty^x$$

$$\phi_\infty^2 = v \sin \theta \sin \phi \quad (u=1 \text{ merp})$$

n Times Around M_0

$$\phi_{\infty}^3 = V \cos \theta$$

$$\phi_{\infty}^1 = V \sin \theta \cos n \phi$$

$$\phi_{\infty}^2 = V \sin \theta \sin n \phi$$

covering the sphere n times

$n=1$ = magnetic monopole

Proof:

$$Q_{em}^{\infty} = La \frac{4\pi}{V}$$

$$A_\mu^\infty = A_\mu^a \chi_a / \gamma$$

$$F_{\mu\nu}^\infty = F_{\mu\nu}^a(\infty) \frac{\chi_a}{\gamma} - \dots$$

$$- \frac{1}{g} \epsilon_{abc} \frac{\phi_\infty^a (D_\mu \phi)_b^\infty (D_\nu \phi)_c^\infty}{|\phi_\infty|^3}$$

neglect $\left(\frac{\dots}{1/r^4} \right)$



$$F_{\mu\nu}^\infty \rightarrow F_{\mu\nu}^a(\infty) \frac{\chi_a}{\gamma}$$

Compute it!

$$(D_i \phi)^\infty = 0$$

$$\Rightarrow \partial_i \phi^\infty - ig (L_a A_a^i) \phi^\infty = 0$$

$$\phi_b^\infty = v \frac{x_b}{\gamma}$$

$$\partial_i \left(\frac{x_b}{\gamma} \right) = ig (L_a)_{bc} A_i^a \cdot \frac{x_c}{v}$$

$$= -g \epsilon_{abc} A_i^a \frac{x_c}{v}$$

$$A_i^a = a \frac{\delta_i^a}{\gamma} + b \epsilon_{iab} \frac{x_b}{\gamma}$$

$$\partial_i \left(\frac{x_b}{\gamma} \right) = \frac{\delta_{ib}}{\gamma} - \frac{x_i x_b}{\gamma^3}$$



$$A_i^a = \frac{1}{g} \epsilon^{iab} \frac{x_b}{r^2} \quad (\pm)$$

Prove!

$$\partial_i \frac{x_b}{r} = -g \epsilon_{abc} A_i^a \frac{x_c}{r} \frac{1}{g}$$

$$= -g \epsilon_{abc} \epsilon^{iad} \frac{x_c x_d}{r^2} \frac{1}{g}$$

$$= +g \epsilon_{abc} \epsilon^{aib} \frac{x_c x_d}{r^2} \frac{1}{g}$$

$$= g (\delta^{ib} \delta^{cd} - \delta^{ci} \delta^{bd}) \frac{x_c x_d}{r^2} \frac{1}{g}$$

$$= \cancel{g} \frac{\delta^{ib} r^2 - x_i x_b}{r^2} \frac{1}{\cancel{g}} \quad \checkmark$$



$$F_{ij}^{\infty} = F_{ij}^a \frac{x_a}{r} =$$

$$= \left(\partial_i A_j^a - \partial_j A_i^a + g \epsilon_{abc} A_i^b A_j^c \right) \frac{x_a}{r}$$

$$= \frac{1}{g} \left(\partial_i \left(\epsilon_{jab} \frac{x_b}{r^2} \right) - \partial_j \left(\epsilon_{iab} \frac{x_b}{r^2} \right) \right.$$

$$\left. + \epsilon_{abc} \epsilon_{ibc} \frac{x_k}{r^2} \epsilon_{jce} \frac{x_e}{r^2} \right) \frac{x_a}{r}$$

$$= \frac{1}{g} \left(\epsilon_{jab} \left(\frac{\delta_{ib}}{r^2} + \dots \right) \frac{x_a}{r} \right.$$

$$\left. - \epsilon_{iab} \frac{\delta_{jb}}{r^2} \frac{x_a}{r} \right.$$

$$\left. + (\delta_{ai} \delta_{ca} - \delta_{aibc}) \epsilon_{jce} \frac{x_k x_e x_a}{r^5} \right)$$

$$= \frac{1}{\rho} \left(-2 \epsilon_{ija} \frac{x_a}{r^3} - \epsilon_{jil} \frac{x_l r^2}{r^5} \right)$$

$$= \frac{1}{\rho} \left(-2 \epsilon_{ija} \frac{x_a}{r^3} + \epsilon_{ije} \frac{x_e}{r^3} \right)$$



$$F_{ij}^{\infty} = -\frac{1}{\rho} \epsilon_{ija} \frac{x_a}{r^3}$$

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$$\epsilon_{ija} B_a^{\infty} \Rightarrow$$

$$\vec{B}_a^{\infty} = \frac{1}{\rho} \frac{\hat{r}}{r^2}$$



Magnetic monopole

$$\vec{B}(j_m) = \frac{j_m}{4\pi r^2} \hat{r}$$

$$\Rightarrow \oint j_m = 4\pi$$

Q. E. D.

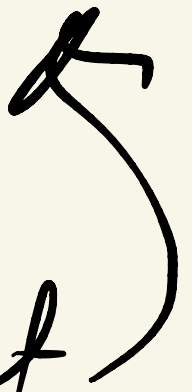
1948 (?)

\exists magnetic monopole

Dirac



Q.N of e in field of



$$\int e^{i g_m} = 2\pi n$$

$$n = 1, 2, \dots$$

$SO(3) \Rightarrow Q_{ew} = \text{quantized}$



$SU(2) \leftarrow \text{Spinor}$