https://moodle.lmu.de → Kurse suchen: 'Rechenmethoden'

Sheet 15: Optional: Frequency Comb

Solution Optional Problem 1: Poisson summation formulas [3]

(a) We multiply the completeness relation with $f(y/L)$ and integrate over $x = y/L$:

Completeness: $$\frac{1}{L}\sum_{n\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}2\pi ny/L} = \sum_{m\in\mathbb{Z}} \delta(y - Lm) = \frac{1}{L}\sum_{m\in\mathbb{Z}} \delta\left(\tfrac{y}{L} - m\right) \tag{1}$$

$\int \mathrm{d}x f(x)(1)$: $$\sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} \mathrm{d}x\, f(x)\mathrm{e}^{-\mathrm{i}2\pi nx} = \sum_{m\in\mathbb{Z}} \int_{-\infty}^{\infty} \mathrm{d}x\, f(x)\delta(x-m) = \sum_{m\in\mathbb{Z}} f(m)\,.$$

The left integral corresponds to the Fourier transform, $\tilde{f}(k) = \int_{-\infty}^{\infty} \mathrm{d}x\, f(x)\mathrm{e}^{-\mathrm{i}kx}$, with $k = 2\pi n$. Thus we obtain:

$$\boxed{\sum_{n\in\mathbb{Z}} \tilde{f}(2\pi n) = \sum_{m\in\mathbb{Z}} f(m)}\,. \tag{2}$$

(b) We set $f(x) = \mathrm{e}^{-a|x|}$, and then calculate the Fourier transform:

$$\tilde{f}(k) = \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}kx} f(x) = \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-(\mathrm{i}kx + a|x|)} = \frac{2a}{k^2 + a^2}\,.$$

The Poisson summation formula $\sum_n \tilde{f}(2\pi n) = \sum_m f(m)$ then gives:

$$\boxed{\sum_{n\in\mathbb{Z}} \frac{2a}{(2\pi n^2 + a^2)}} = \sum_{m\in\mathbb{Z}} \mathrm{e}^{-a|m|} = 2\sum_{m=0}^{\infty} \mathrm{e}^{-am} - 1$$
$$= \frac{2}{1 - \mathrm{e}^{-a}} - 1 = \frac{1 + \mathrm{e}^{-a}}{1 - \mathrm{e}^{-a}} = \frac{\mathrm{e}^{a/2} + \mathrm{e}^{-a/2}}{\mathrm{e}^{a/2} - \mathrm{e}^{-a/2}} = \boxed{\coth(a/2)}\,.$$

(c) We set $f(x) = \mathrm{e}^{-(ax^2 + bx + c)}$, and then calculate the Fourier transform:

$$\tilde{f}(k) = \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}kx} f(x) = \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-a\left(x^2 + \frac{1}{a}(b+\mathrm{i}k)x\right) - c}$$
$$= \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-a\left(x + \frac{b+\mathrm{i}k}{2a}\right)^2} \mathrm{e}^{a\left(\frac{b+\mathrm{i}k}{2a}\right)^2 - c} = \sqrt{\frac{\pi}{a}}\mathrm{e}^{\frac{1}{4a}\left(b^2 + 2\mathrm{i}bk - k^2\right) - c}\,.$$

The Poisson summation formula $\sum_m f(m) = \sum_n \tilde{f}(2\pi n)$ then gives:

$$\boxed{\sum_{m\in\mathbb{Z}} \mathrm{e}^{-\left(am^2+bm+c\right)} = \sqrt{\frac{\pi}{a}}\mathrm{e}^{\left(\frac{b^2}{4a}-c\right)} \sum_{n\in\mathbb{Z}} \mathrm{e}^{-\frac{1}{a}\left(\pi^2 n^2 + \mathrm{i}\pi n b\right)}}\,.$$

Solution Optional Problem 2: Fourier integral representation of a periodic function, frequency comb [6]

(a) We insert the Fourier series for $p(t)$ into the formula for the Fourier transform of $p(t)$:

$$\tilde{p}(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t}\, p(t) = \frac{1}{\tau}\sum_m \underbrace{\int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t}\mathrm{e}^{-\mathrm{i}\omega_m t}}_{2\pi\delta(\omega-\omega_m)}\, \tilde{p}_m = \boxed{\omega_r \sum_m \tilde{p}_m \delta(\omega-\omega_m)}\,,$$

with $\omega_r = 2\pi/\tau$. We now clearly see that $\tilde{p}(\omega)$ is a periodic frequency comb of δ functions, whose weights are fixed by the coefficients $\tilde{p}_m$ of the Fourier series.

(b) We insert the Fourier representation, $f(t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{-\mathrm{i}\omega t}\tilde{f}(\omega)$, into the definition of $p(t)$ and then perform the substitution $\omega = y\omega_r$ (implying $\omega\tau = 2\pi y$):

$$\begin{aligned} p(t) &= \sum_n f(t-n\tau) = \sum_n \int_{-\infty}^{\infty} \tfrac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{-\mathrm{i}\omega(t-n\tau)}\tilde{f}(\omega) \\ &\overset{\omega=y\omega_r}{=} \sum_n \int_{-\infty}^{\infty} \mathrm{d}y\, \mathrm{e}^{\mathrm{i}2\pi y n} \underbrace{\left[\mathrm{e}^{-\mathrm{i}y\omega_r t}\tfrac{1}{\tau}\tilde{f}(y\omega_r)\right]}_{\equiv F(y)} = \sum_n \tilde{F}(2\pi n) \overset{\text{(Poisson)}}{=} \sum_m F(m) \end{aligned}$$

Here we have defined the function $F(y) = \mathrm{e}^{-\mathrm{i}y\omega_r t}\frac{1}{\tau}\tilde{f}(y\omega_r)$, with Fourier transform $\tilde{F}(k)$, and used the Poisson summation formula. Using $\omega_m = m\omega_r = 2\pi m/\tau$, we thus obtain:

$$p(t) = \sum_m F(m) = \frac{1}{\tau}\sum_m \mathrm{e}^{-\mathrm{i}m\omega_r t}\underbrace{\tilde{f}(m\omega_r)}_{\equiv\tilde{p}_m} \overset{\omega_m=m\omega_r}{=} \frac{1}{\tau}\sum_m \mathrm{e}^{-\mathrm{i}\omega_m t}\tilde{p}_m \quad \text{with} \quad \boxed{\tilde{p}_m = \tilde{f}(\omega_m)}\,.$$

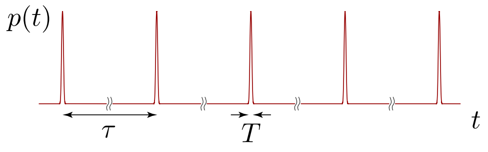
The middle term has the form of a discrete Fourier series, from which we can read off the discrete Fourier coefficients $\tilde{p}_m$ of $p(t)$. They are clearly given by $\tilde{p}_m = \tilde{f}(\omega_m)$, and correspond to the Fourier transform of $f(t)$ evaluated at the discrete frequencies ω_m.

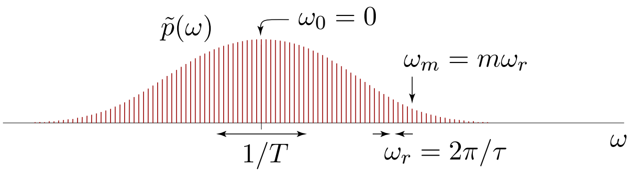
(c) From (a) and (b) we directly obtain the following form for the Fourier transform of $p(t)$:

Fourier spectrum: $$\tilde{p}(\omega) \overset{\text{(a)}}{=} \omega_r \sum_m \tilde{p}_m\delta(\omega-\omega_m) \overset{\text{(b)}}{=} \boxed{\omega_r \sum_m \tilde{f}(\omega_m)\delta(\omega-\omega_m)}\,.$$

This is a comb of peaks at equally spaced frequencies ω_m, with weights governed by the envelope function $\tilde{f}(\omega_m)$. For a series of Gaussian functions, $p_G(t) = \sum_n f_G(t-n\tau)$, the corresponding envelope, $\tilde{f}_G(\omega)$, has the form of a Gaussian, too (Sheet 12, Example Problem 2):

Envelope: $$\tilde{f}_G(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} f_G(t) = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} \frac{1}{\sqrt{2\pi T^2}} \mathrm{e}^{-\frac{t^2}{2T^2}} = \boxed{\mathrm{e}^{-\frac{1}{2}T^2\omega^2}}.$$





(d) The Fourier transform of $E(t) = \mathrm{e}^{-\mathrm{i}\omega_c t} p(t)$, to be denoted $\tilde{E}(\omega)$, is the same as that of $p(t)$, except that the frequency argument is shifted by ω_c:

$$\tilde{E}(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} E(t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega'}{2\pi} \tilde{p}(\omega') \underbrace{\int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}(\omega - \omega_c - \omega')t}}_{2\pi\delta(\omega - \omega_c - \omega')} = \tilde{p}(\omega - \omega_c)$$

$$\overset{\text{(c)}}{=} \frac{2\pi}{\tau} \sum_m \tilde{f}(\omega_m)\delta(\omega - \omega_m - \omega_c) \overset{m=n-N}{=} \boxed{\frac{2\pi}{\tau} \sum_n \tilde{f}(\omega_{n-N})\delta(\omega - \omega_n - \omega_{\text{off}})}.$$

For the last step we used $\omega_c = N\omega_r + \omega_{\text{off}}$ and renamed the summation index, $m = n - N$, such that $\omega_m + \omega_c = \omega_n + \omega_{\text{off}}$. Thus $\tilde{E}(\omega)$ forms an 'offset-shifted' frequency comb, whose peaks relative to the Fourier frequencies ω_n are shifted by the offset frequency ω_{off}. The 'center' of the comb lies at the frequency where $\tilde{f}(\omega_{n-N})$ is maximal, i.e. at $n = N$, with frequency $\omega_N \simeq \omega_c$.

(e) We begin with the definition of the Fourier transform of $p_\gamma(t)$:

Definition: $$\tilde{p}_\gamma(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} p_\gamma(t) = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} \sum_n f(t - n\tau)\mathrm{e}^{-|n|\tau\gamma}$$

$t' = t - n\tau$: $$= \underbrace{\sum_n \mathrm{e}^{\mathrm{i}n\tau\omega}\mathrm{e}^{-|n|\tau\gamma}}_{\equiv S^{[\gamma,\omega_r]}(\omega)} \underbrace{\int_{-\infty}^{\infty} \mathrm{d}t'\, \mathrm{e}^{\mathrm{i}\omega t'} f(t')}_{=\tilde{f}(\omega)} . \tag{3}$$

The sum

$$S^{[\gamma,\omega_r]}(\omega) \equiv \sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}n\tau\omega}\mathrm{e}^{-|n|\tau\gamma} \overset{\tau=2\pi/\omega_r}{=} \sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}2\pi n\omega/\omega_r}\mathrm{e}^{-2\pi|n|\gamma/\omega_r} \tag{4}$$

has the same form as a damped sum over Fourier modes,

$$S^{[\epsilon,L]}(x) \equiv \sum_{k\in\frac{2\pi}{L}\mathbb{Z}} \mathrm{e}^{\mathrm{i}kx - \epsilon|k|} = \sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}2\pi nx/L}\mathrm{e}^{-2\pi|n|\epsilon/L} . \tag{5}$$

The latter can summed using geometric series in the variables $\mathrm{e}^{-2\pi(\epsilon\mp\mathrm{i}x)/L}$ (sheet 11, Homework Problem 3)]:

$$S^{[\epsilon,L]}(x) = \frac{1-\mathrm{e}^{-4\pi\epsilon/L}}{1+\mathrm{e}^{-4\pi\epsilon/L}-2\mathrm{e}^{-2\pi\epsilon/L}\cos(2\pi x/L)} \simeq L\sum_{m\in\mathbb{Z}}\delta_{\mathrm{LP}}^{[\epsilon]}(x-mL)\,. \tag{6}$$

The result is a periodic sequence of peaks at the positions $x \simeq mL$, each with the form of a Lorentzian peak (LP), $\delta_{\mathrm{LP}}^{[\epsilon]}(x) = \frac{\epsilon/\pi}{x^2+\epsilon^2}$ for $x, \epsilon \ll L$. Using the association $x \mapsto \omega$, $\epsilon \mapsto \gamma$ and $L \mapsto \omega_r$ we obtain:

$$S^{[\gamma,\omega_r]}(\omega) \overset{(4,6)}{=} \omega_r \sum_{m\in\mathbb{Z}} \delta_{\mathrm{LP}}^{[\gamma]}(\omega - m\omega_r) \tag{7}$$

and

$$\tilde{p}_\gamma(\omega) \overset{(3,7)}{=} \boxed{\omega_r \sum_{m\in\mathbb{Z}} \delta_{\mathrm{LP}}^{[\gamma]}(\omega-\omega_m)\tilde{f}(\omega)}\,.$$

Thus the spectrum of a series of periodic pulses, truncated beyond $|n| \lesssim 1/(\tau\gamma)$, corresponds to a frequency comb with Lorentz-broadened peaks as teeth, each with width $\simeq \gamma$.

[Total Points for Optional Problems: 9]
