https://moodle.lmu.de → Kurse suchen: 'Rechenmethoden'

Sheet 15: Optional: Frequency Comb

(b)[2](E/M/A) means: problem (b) counts 2 points and is easy/medium hard/advanced

Optional Problem 1: Poisson summation formulas [3]
Points: (a)[1](M); (b)[1](E); (c)[1](M). [T]

(a) Show that every function $f(x)$, for which a Fourier-integral representation of the form $f(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi}\, \mathrm{e}^{\mathrm{i}kx} \tilde{f}(k)$ exists, fulfills the following remarkable relationship:

'Poisson summation formula': $$\sum_{m\in\mathbb{Z}} f(m) = \sum_{n\in\mathbb{Z}} \tilde{f}(2\pi n)\,.$$

The sum of the function values $f(m)$ over all the integers is exactly the same as the sum over all the Fourier coefficients $\tilde{f}(2\pi n)$!

Hint: multiply the completeness relation for discrete Fourier modes, $\frac{1}{L}\sum_{n\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}2\pi n y/L} = \sum_{m\in\mathbb{Z}} \delta(y - Lm)$, with $f(y/L)$ then integrate over $x = y/L$.

Using the Poisson summation formula and the following functions $f(x)$, prove the following identities (with $0 < a \in \mathbb{R}$):

(b) $f(x) = \mathrm{e}^{-a|x|}$: $$\sum_{n\in\mathbb{Z}} \frac{2a}{(2\pi n)^2 + a^2} = \coth(a/2)\,.$$

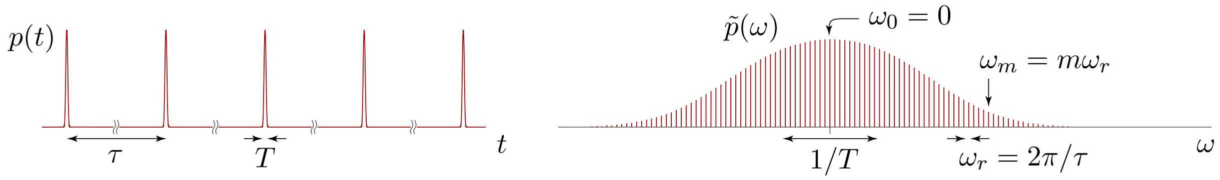
(c) $f(x) = \mathrm{e}^{-(ax^2+bx+c)}$: $$\sum_{m\in\mathbb{Z}} \mathrm{e}^{-(am^2+bm+c)} = \sqrt{\frac{\pi}{a}}\,\mathrm{e}^{\left(\frac{b^2}{4a}-c\right)} \sum_{n\in\mathbb{Z}} \mathrm{e}^{-\frac{1}{a}\left(\pi^2 n^2 + \mathrm{i}\pi n b\right)}\,.$$

The identity (c) is the so-called 'Poisson resummation formula' for infinite sums over discrete Gaussian functions. Note that this is an example of Fourier reciprocity: the width of the discrete Gaussian functions on the left and right hand sides of the equation are proportional to $1/a$ and a/π^2 respectively.

Optional Problem 2: Fourier integral representation of a periodic function, frequency comb [6]
Points: (a)[1](E); (b)[2](M); (c)[1](E); (d)[2](M); (e)[3,Bonus](A).

The most precise method we have today of measuring optical frequencies employs an optical 'frequency comb'. John L. Hall (University of Colorado, USA) and Theodor W. Hänsch (Ludwig-Maximilians-University Munich) shared the Nobel prize in physics 2005 for the development of this technology.
An optical cavity is used to generate a precisely periodic series of light pulses, whose Fourier spectrum has the form of a frequency comb. This comb has more than a million 'teeth'; they span more than an octave and their frequencies are known to a precision of better than one part in 10^{15}. To measure the frequency of a beam of light in the optical spectrum, it is superimposed

with the frequency comb. Whichever tooth is closest in frequency to the frequency to be measured will combine with the latter to form a 'beat' in the radio frequency range, allowing the unknown frequency to be determined to better than one part in 10^{15}.
In the following, we describe the mathematical principles underlying the frequency comb technique. The main idea is simple: the Fourier transform of a periodic signal yields a periodic δ-function — the frequency comb. We are interested in finding the positions, heights and widths of the comb's 'teeth', and how these change when the input signal is not perfectly periodic, as is bound to be the case in the lab.



Let $p(t)$ be a periodic function with period τ. It has a Fourier series representation, $p(t) = \frac{1}{\tau}\sum_{m\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}\omega_m t}\,\tilde{p}_m$, whose discrete Fourier frequencies, $\omega_m = m\omega_r$, are multiples of $\omega_r = 2\pi/\tau$. The same function also has a Fourier integral representation, $p(t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-\mathrm{i}\omega t}\tilde{p}(\omega)$.

(a) Show that the Fourier transform $\tilde{p}(\omega)$ is given by a sum of discrete δ functions — the 'frequency comb' — whose teeth coincide with the discrete Fourier frequencies ω_m, weighted with the discrete Fourier coefficients $\tilde{p}_m$:

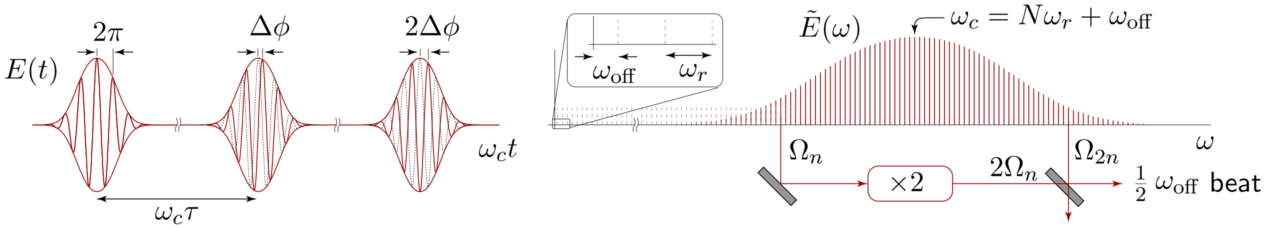
$$\tilde{p}(\omega) = \omega_r \sum_m \tilde{p}_m\, \delta(\omega - \omega_m). \tag{1}$$

We henceforth consider a periodic function of the form $p(t) \equiv \sum_{n\in\mathbb{Z}} f(t-n\tau)$, consisting of a series of shifted versions of the same 'seed function', $f(t)$. For example, if the seed function describes a single peak, the periodic function describes a train of such peaks. Let $f(t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-\mathrm{i}\omega t}\tilde{f}(\omega)$ denote the Fourier integral representation of the seed function.

(b) Show that the coefficients $\tilde{p}_m$ of the Fourier series representation of $p(t)$ are determined by the Fourier transform of the seed function via $\tilde{p}_m = \tilde{f}(\omega_m)$. *Hint:* Insert the Fourier integral representation of the seed function $f(t)$ into the definition of $p(t)$. Then use the Poisson summation formula, $\sum_m \tilde{F}(2\pi m) = \sum_n F(n)$ (see corresponding example problem), to bring $p(t)$ into the form of a Fourier series. Read off the Fourier coefficients $\tilde{p}_m$ from the latter.

To be concrete, let us now consider a seed function having the form of a Gaussian peak, $f_G(t) = \frac{1}{\sqrt{2\pi T^2}}\mathrm{e}^{-t^2/(2T^2)}$. Let $p_G(t)$ denote a periodic train of such peaks. Moreover, we take the Gaussian peak width, T, to be much smaller than the period of the periodic train, $T \ll \tau$.

(c) According to Eq. (1), the Fourier transform, $\tilde{p}_G(\omega)$, of the Gaussian train constitutes a frequency comb. Find a formula for $\tilde{p}_G(\omega)$ and identify the positions and weights of the comb's peaks. Show that the peak weights are governed by a Gaussian envelope whose width is inversely proportional to that of the Gaussian seed function — Fourier reciprocity!



An optical frequency comb generator emits a beam of light, whose electric field (which we present in a simplified manner) has the form $E(t) = \mathrm{e}^{-\mathrm{i}\omega_c t} p(t)$. Here $\mathrm{e}^{-\mathrm{i}\omega_c t}$ represents the oscillation of a 'carrier signal', with carrier frequency ω_c. It is modulated by a pulse train $p(t)$, a precisely periodic series of short pulses (with pulse duration/period $= T/\tau \ll 10^{-6}$). The pulse train and the carrier signal are typically 'incommensurate': the carrier frequency ω_c typically lies *between* two teeth (rather than *on* a tooth) of the frequency comb generated by the pulse train. It thus has the generic form $\omega_c = N\omega_r + \omega_{\text{off}}$, with an 'offset frequency' $\omega_{\text{off}} \in (0, \omega_r)$. Due to this incommensurability, the combined signal $E(t)$ is not periodic: the phase of the carrier signal relative to the pulse train shifts or 'slips' from one pulse to the next by an amount $\Delta\phi = \omega_{\text{off}}\tau < 2\pi$.

(d) Show that the Fourier spectrum of $E(t)$ forms a frequency comb, whose 'center' has been shifted from 0 to ω_c, and whose teeth are shifted relative to the Fourier frequencies ω_n by the offset frequency ω_{off}:

$$\tilde{E}(\omega) = \omega_r \sum_n \tilde{f}(\omega_{n-N}) \delta(\omega - \omega_n - \omega_{\text{off}}) \,. \tag{2}$$

Precise frequency measurements using a frequency comb require accurate knowledge of the teeth positions, $\Omega_n = n\omega_r + \omega_{\text{off}}$, and hence also of ω_r and ω_{off}. The frequency $\omega_r = 2\pi/\tau$ is typically very stable, but ω_{off} undergoes slow, irregular fluctuations as function of time. Trying to measure frequencies with such a comb would be like trying to measure distances with a shaking ruler. The key insight, due to Hänsch, that made it possible to 'control this shaking' is that ω_{off} can be measured accurately if the teeth span at least a full octave. In this case, a tooth near the lower end of the comb, with frequency Ω_n, has the property that twice its frequency again lies within the range of the comb. By doubling Ω_n (a standard procedure in optics) and superimposing the resulting signal with the comb, one thus obtains beats between $2\Omega_n$ and a tooth of frequency Ω_{2n}. The beat frequency is $\frac{1}{2}(2\Omega_n - \Omega_{2n}) = \frac{1}{2}\left[2(n\omega_r + \omega_{\text{off}}) - (2n\omega_r + \omega_{\text{off}})\right] = \frac{1}{2}\omega_{\text{off}}$. By observing the beat signal one may thus monitor ω_{off} and stabilize its value via a feedback loop. This ultimately allows a frequency comb to be stabilized to a precision of one part in 10^{15}.

So far, we have assumed the pulse train $p(t)$ to be strictly periodic. Deviations from perfect periodicity cause the teeth of the frequency comb to be broadened. To illustrate this, let us consider a 'truncated' pulse train existing only for a finite amount of time, of the form $p_\gamma(t) = \sum_{n\in\mathbb{Z}} f(t - n\tau)\mathrm{e}^{-|n|\tau\gamma}$, with $\tau\gamma \ll 1$. The factor $\mathrm{e}^{-|n|\tau\gamma}$ suppresses the contributions for which $|n| \gtrsim 1/(\tau\gamma) \gg 1$, thereby 'truncating' the series.

(e) Compute the Fourier transform $\tilde{p}_\gamma(\omega)$ of the truncated pulse train. What shape do the individual teeth have? Show that their width is inversely proportional to the 'duration' of the truncated pulse chain — Fourier reciprocity again! *Hint:* To compute $\tilde{p}_\gamma(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} p_\gamma(t)$, express $p_\gamma(t)$ in terms of the Fourier representation of $f(t)$. Use the substitution $t' = t - n\tau$ in the time integral to arrive at an expression of the form $\tilde{p}_\gamma(\omega) = S(\omega)\tilde{f}(\omega)$, where $S(\omega)$ is given by a geometries series. Evaluate this series and analyze the shape of the peak at $\omega \simeq m\omega_r$ in the limit $\gamma\tau \ll 1$.

To summarize, we have encountered the following general relationships, which, remarkably, all come into play in the frequency comb measuring technique: (a) A periodic function $p(t)$ has a discrete Fourier *series* representation, with discrete Fourier frequencies ω_n. Therefore its Fourier *integral* representation, $\tilde{p}(\omega)$, must consist of a series of δ functions at these discrete frequencies — forming a frequency comb. (b) For a periodic function of the form $p(t) = \sum_n f(t-n\tau)$, where $f(t)$ is some seed function, the envelope of the frequency comb corresponds to the Fourier transform of the seed function, $\tilde{p}_m = \tilde{f}(\omega_m)$. (c) Fourier reciprocity applies: if the seed function $f(t)$ describes a peak, then the narrower this peak, the broader the peak described by its Fourier transform, $\tilde{f}(\omega_m)$, and hence the broader the envelope of the frequency comb. (d) When the periodic function $p(t)$ is multiplied with a periodic carrier signal whose frequency is incommensurate with that of the comb, then the comb is shifted by an offset frequency. (e) If $p(t)$ is truncated to lie within some bounded time interval, then the teeth of the frequency comb are broadened — Fourier reciprocity again.

[Total Points for Optional Problems: 9]
