Sheet 10: Differential Equations II. Asymptotic Expansions

Posted: Mo 20.12.21 Central Tutorial: Th 23.12.21 Due: Th 13.01.22, 14:00
(b)[2](E/M/A) means: problem (b) counts 2 points and is easy/medium hard/advanced
Suggestions for central tutorial: example problems 2, 3, 4, 5.
Videos exist for example problems 3 (C7.4.7), 5 (C5.4.1).

Example Problem 1: Substitution and separation of variables [2]
Points: (a)[1](E); (b)[1](E).

Often differential equations can be solved by a suitably chosen substitution.

(a) Consider the differential equation $y' = f(y/x)$ for the function $y(x)$. Show that the substitution $y = ux$ can be used to convert it into a separable differential equation for the function $u(x)$, which can be solved using separation of variables.

(b) Use this method to solve the equation $xy' = 2y + x$ with the initial condition $y(1) = 0$. [Check your result: $y(2) = 2$.]

Example Problem 2: Inhomogeneous linear differential equation, variation of constant [3]
Points: (a)[1](E); (b)[2](M).

Solve the inhomogeneous differential equation $\dot{x} + 2x = t$ with $x(0) = 0$, as follows:

(a) Determine the general solution of the homogeneous equation.

(b) Then find a special (particular) solution to the inhomogeneous problem by means of variation of constants. [Check your result: $x(-\ln 2) = \frac{3}{4} - \frac{1}{2}\ln 2$.]

Example Problem 3: Inhomogeneous linear differential equation of second order: driven overdamped harmonic oscillator [7]
Points: (a)[1](E); (b)[2](M); (c)[2](M); (d)[2](M).

Consider the following driven, over-damped harmonic oscillator with $\gamma > \Omega$:

Differential equation: $$\ddot{x} + 2\gamma\dot{x} + \Omega^2 x = f_A(t). \tag{1}$$

Initial values: $$x(0) = 0, \quad \dot{x}(0) = 1\ , \tag{2}$$

Driving function: $$f_A(t) = \begin{cases} f_A & \text{for} \quad t \geq 0, \\ 0 & \text{for} \quad t < 0. \end{cases}$$

For $t > 0$, find a solution to this equation of the form $x(t) = x_h(t) + x_p(t)$, where the homogeneous solution, $x_h(t)$, solves the homogeneous DEQ, with initial values (2), and the particular solution, $x_p(t)$, solves the inhomogeneous DEQ, with initial values $x_p(0) = \dot{x}_p(0) = 0$. Proceed as follows:

(a) Rewrite as matrix equation: Write the DEQ (1) in the matrix form

$$\dot{\mathbf{x}} = A \cdot \mathbf{x} + \mathbf{b}(t), \quad \text{with} \quad \mathbf{x} \equiv (x, \dot{x})^T \equiv (x^1, x^2)^T \, . \tag{3}$$

Find the matrix A, the driving force vector $\mathbf{b}(t)$, and the initial value $\mathbf{x}_0 = \mathbf{x}(0)$.

(b) Homogeneous solution: Find the solution, $\mathbf{x}_h(t)$, of the homogeneous DEQ $(3)|_{\mathbf{b}(t)=0}$ having the initial value $\mathbf{x}_h(0) = \mathbf{x}_0$. Use the ansatz $\mathbf{x}_h(t) = \sum_j c_h^j \mathbf{x}_j(t)$, with $\mathbf{x}_j(t) = \mathbf{v}_j e^{\lambda_j t}$, where λ_j and $\mathbf{v}_j$ $(j = 1, 2)$ are the eigenvalues and the eigenvectors of A. What does the corresponding solution, $x_h(t) = {x^1}_h(t)$, of the homogeneous differential equation $(1)|_{f_A(t)=0}$ look like? [Check your result: if $\gamma = \sqrt{2}\ln 2$ and $\Omega = \ln 2$, then $x_h(1) = \frac{3}{4}\frac{2-\sqrt{2}}{\ln 2}$.]

(c) Particular solution: Using the ansatz $\mathbf{x}_p(t) = \sum_j c_p^j(t)\mathbf{x}_j(t)$ (variation of constants), find the particular solution of the inhomogeneous differential equation (3) having the initial value $\mathbf{x}_p(0) = \mathbf{0}$. What is the corresponding solution, $x_p(t) = {x^1}_p(t)$, of the inhomogeneous DEQ (1)? [Check your result: if $\gamma = 3\ln 2$, $\Omega = \sqrt{5}\ln 2$ and $f_A = 1$, then $x_p(1) = \frac{49}{640}\frac{1}{(\ln 2)^2}$.]

(d) Qualitative discussion: The desired solution of the inhomogeneous DEQ (1) is given by $x(t) = x_h(t) + x_p(t)$. Sketch your result for this function qualitatively for the case $f_A < 0$, and explain the behavior as $t \to 0$ and $t \to \infty$.

Example Problem 4: System of linear differential equations with non-diagonizable matrix [4]

Points: (a)[1](E); (b)[1](M); (c)[1](A); (d)[0.5](E); (e)[0.5](E).

We consider a procedure to solve the differential equation

$$\dot{\mathbf{x}} = A \cdot \mathbf{x} \tag{4}$$

for the case of a matrix $A \in \mathrm{Mat}(n, \mathbb{R})$ that has $n-1$ distinct eigenvalues λ_j and associated eigenvectors $\mathbf{v}_j$, with $j = 1, \ldots, n-1$, where the eigenvalue $\lambda_{n-1} = \lambda_n$ is a two-fold zero of the characteristic polynomial but has only *one* eigenvector. Such a matrix is not diagonalizable. However, it can be brought into the so-called Jordan normal form:

$$T^{-1}AT = J, \qquad J = \begin{pmatrix} \lambda_1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \cdots & \cdots & \lambda_{n-1} & 1 \\ 0 & \cdots & \cdots & 0 & \lambda_{n-1} \end{pmatrix}, \qquad T = (\mathbf{v}_1, \cdots, \mathbf{v}_{n-1}, \mathbf{v}_n) \, . \tag{5}$$

Using $A = TJT^{-1}$, as well as $\mathbf{v}_j = T\mathbf{e}_j$ and $J\mathbf{e}_j = \lambda_j \mathbf{e}_j + \delta_{jn}\mathbf{e}_{j-1}$, one finds that this is equivalent to

$$A \cdot \mathbf{v}_j = \lambda_j \mathbf{v}_j + \mathbf{v}_{j-1}\delta_{jn}, \quad \forall j = 1, \ldots, n. \tag{6}$$

For $j = 1, \ldots, n-1$ this corresponds to the usual eigenvalue equation, and $\mathbf{v}_j$ to the usual eigenvectors. $\mathbf{v}_n$, however, is not an eigenvector, but is rather determined by the following equation:

$$(A - \mathbb{1}\lambda_n)\mathbf{v}_n = \mathbf{v}_{n-1}. \tag{7}$$

Since $(A - \mathbb{1}\lambda_n)$ is not invertible, this equation does not uniquely fix the vector $\mathbf{v}_n$. Different choices of $\mathbf{v}_n$ lead (via (5)) to different similarity transformation matrices T, but they all yield the same form for the Jordan-Matrix J.
The λ_j and $\mathbf{v}_j$ thus obtained can be used to find a solution for the DEQ (4), using an exponential ansatz together with 'variation of the constants':

$$\mathbf{x}(t) = \sum_{j=1}^{n} \mathbf{v}_j \mathrm{e}^{\lambda_j t} c^j(t), \quad \text{with} \quad \lambda_n \equiv \lambda_{n-1}. \tag{8}$$

The coefficients $c^j(t)$ can be determined by inserting this ansatz into (4):

$$0 = \left(\tfrac{\mathrm{d}}{\mathrm{d}t} - A\right)\mathbf{x}(t) = \sum_{j=1}^{n} \mathbf{v}_j \mathrm{e}^{\lambda_j t}\left[\lambda_j c^j(t) + \dot{c}^j(t) - \lambda_j c^j(t)\right] - \mathbf{v}_{n-1}\mathrm{e}^{\lambda_n t} c^n(t). \tag{9}$$

Comparing coefficients of $\mathbf{v}_j$ we obtain:

$$\mathbf{v}_{j \neq n-1}: \qquad \dot{c}^j(t) = 0 \qquad \Rightarrow \qquad \boxed{c^j(t) = c^j(0) = \text{const.}}\,, \tag{10}$$

$$\mathbf{v}_{n-1}: \qquad \dot{c}^{n-1}(t) = c^n(t) \qquad \Rightarrow \qquad \boxed{c^{n-1}(t) = c^{n-1}(0) + t\, c^n(0)}\,. \tag{11}$$

The values of $c^j(0)$ are fixed by the initial conditions $\mathbf{x}(0)$:

$$\mathbf{x}(0) = \sum_j \mathbf{v}_j c^j(0) = T\mathbf{c}(0)\,, \quad \Rightarrow \quad \mathbf{c}(0) = T^{-1}\mathbf{x}(0)\,. \tag{12}$$

Now use this method to find the solution of the DEQ

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad \text{with} \quad A = \frac{1}{3}\begin{pmatrix} 7 & 2 & 0 \\ 0 & 4 & -1 \\ 2 & 0 & 4 \end{pmatrix} \quad \text{and} \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{13}$$

(a) Show that the characteristic polynomial for A has a simple zero, say λ_1, and a two-fold zero, say $\lambda_2 = \lambda_3$. [Check: do your results satisfy $\sum_j \lambda_j = \mathrm{Tr}\, A$ and $\prod_j \lambda_j = \det A$?]

(b) Show that the eigenspaces associated with λ_1 and λ_2 are both one-dimensional (which implies that A is not diagonalizable), and find the corresponding normalized eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

(c) Use Eq. (7) to find a third, normalized vector $\mathbf{v}_3$, having the property that A is brought into a Jordan normal form using $T = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$. While doing so, exploit the freedom of choice that is available for $\mathbf{v}_3$ to choose the latter orthonormal to $\mathbf{v}_1$ and $\mathbf{v}_2$. [*Remark:* For the present example orthonormality is achievable (and useful, since then $T^{-1} = T^T$ holds), but this is generally not the case.]

(d) Now use an ansatz of the form (8) to find the solution $\mathbf{x}(t)$ to the DEQ (13). [Check your result: $\mathbf{x}(\ln 2) = (2, 4, 0)^T + \frac{4}{3}(1 + \ln 2)(2, -1, 2)^T$.]

(e) Check your result explicitly by verifying that it satisfies the DEQ.

Example Problem 5: Series expansion for iteratively solving an equation [2]
Points: (a)[1](M); (b)[1](M).

Solve the equation $\mathrm{e}^{y-1} = 1 - \epsilon y$ for y, to second order in the small parameter ϵ, using the ansatz $y(\epsilon) = y_0 + y_1\epsilon + \frac{1}{2!}y_2\epsilon^2 + \mathcal{O}(\epsilon^3)$. Use both of the following approaches:

(a) Method 1: **expansion of equation**. Insert the ansatz for $y(\epsilon)$ into the given equation, Taylor-expand each term to order $\mathcal{O}(\epsilon^2)$, and collect terms having the same power of ϵ to obtain an equation of the form $0 = \sum_n F_n \epsilon^n$. The coefficient of each ϵ^n must vanish, yielding a hierarchy of equations, $F_n = 0$. Starting from $n = 0$, solve these successively for the y_n's, using knowledge of the previously determined $y_{i<n}$ at each step. [Check your results: $y_2 = 1$.]

(b) Method 2: **repeated differentiation**. Method 1 can be viewed from the following perspective: the given equation is written in the form $0 = \mathcal{F}(y(\epsilon), \epsilon) \equiv F(\epsilon)$, and the r.h.s. is brought into the form $\sum_n F_n \epsilon^n$. The latter process can be streamlined by realizing that $F_n = \frac{1}{n!}\mathrm{d}^n_\epsilon F(\epsilon)|_{\epsilon=0}$. Hence, the nth equation in the hierarchy, $F_n = 0$, can be set up by simply differentiating the given equation n times and then setting ϵ to zero, $0 = \mathrm{d}^n_\epsilon F(\epsilon)|_{\epsilon=0}$. Use this approach to find a hierarchy of equations for y_0, y_1 and y_2.

Hint: Since $F(\epsilon)$ depends on ϵ both directly and via $y(\epsilon)$, the chain rule must be used when computing derivates, e.g. $\mathrm{d}_\epsilon F(\epsilon) = \partial_y \mathcal{F}(y, \epsilon) y' + \partial_\epsilon \mathcal{F}(y, \epsilon)$.

Remark: Method 2 has the advantage that it systematically proceeds order by order: information from $\mathcal{O}(\epsilon^n)$ is generated at just the right time, namely when it is needed in step n for computing y_n. As a result, this method is often more convenient than method 1, particularly if the dependence of $\mathcal{F}(y, \epsilon)$ on y is non-trivial.

Example Problem 6: Taylor expansions in two dimensions [2]
Points: (a)[1](E); (b)[1](M).

Find the Taylor expansion of the function $g(x, y) = \mathrm{e}^x \cos(x + 2y)$ in x and y, around the point $(x, y) = (0, 0)$. Calculate explicitly all terms up to and including second order,

(a) by multiplying out the series expansions for the exponential and cosine functions;

(b) by using the formula for the Taylor series of a function of two variables.

[Check your results: the mixed second-order term in each case is: (a) $-2xy$, (b) $-2xy$.]

Example Problem 7: Intersecting planes: minimal distance to origin [2]
Points: [2](M).

Consider the line of intersection of the two planes defined by the equations $x + y + z = 1$ and $x - y + 2z = 2$, respectively. Use Lagrange multipliers to find the point on this line lying closest to the origin. [Check your result: its distance to the origin is $\sqrt{5/7}$.]

[Total Points for Example Problems: 22]

Homework Problem 1: Substitution and separation of variables [7]
Points: (a)[1](E); (b)[1](E); (c)[2](M); (d)[1](E); (e)[2](M).

Consider differential equations of the type

$$y'(x) = f(ax + by(x) + c)\,. \tag{14}$$

(a) Substitute $u(x) = ax + by(x) + c$ and find a differential equation for $u(x)$.

(b) Find an implicit expression for the solution $u(x)$ of the new differential equation using an integral that contains the function f. *Hint:* Separation of variables!

(c) Use the substitution strategy of (a,b) to solve the differential equation $y'(x) = \mathrm{e}^{x+3y(x)+5}$, with initial condition $y(0) = 1$.
[Check your result: $y(\ln(e^{-8}+3) - 2\ln 2) = \frac{1}{3}(2\ln 2 - \ln(e^{-8}+3) - 5)$.]

(d) Check: Solve the differential equation given in (c) directly (without substitution) using separation of variables. Is the result in agreement with the result from (c)?

(e) Solve the differential equation $y'(x) = [a(x+y)+c]^2$ with initial condition $y(x_0) = y_0$ using the substitution given in (a).
[Check your result: if $x_0 = y_0 = 0$ and $a = c = 1$, then $y(0) = 0$.]

Homework Problem 2: Inhomogeneous linear differential equation, variation of constants [2]
Points: (a)[1](E); (b)[1](E); (c)[1,Bonus](E)

The function $x(t)$ satisfies the inhomogeneous differential equation

$$\dot{x}(t) + tx(t) = \mathrm{e}^{-\frac{t^2}{2}}, \qquad \text{with initial condition} \quad x(0) = x_0. \tag{15}$$

(a) Find the solution, $x_h(t)$, of the corresponding homogeneous equation with $x_h(0) = x_0$.

(b) Find the particular solution, $x_p(t)$, of the inhomogeneous equation (15), with $x_p(0) = 0$ using variation of constants, $x_p(t) = c(t)x_h(t)$. What is the general solution?
[Check your result: if $x_0 = 0$, then $x(1) = e^{-1/2}$.]

(c) For a differential equation of the form $\dot{x}(t) + a(t)x(t) = b(t)$ (ordinary, first-order, linear and inhomogeneous), the sum of the homogeneous and inhomogeneous solutions has the form:

$$x(t) = x_h(t) + x_p(t) = x_h(t) + c(t)x_h(t) = (1 + c(t))x_h(t) = \tilde{c}(t)x_h(t)\,.$$

The initial condition $x(0) = x_0$ can therefore also be satisfied by imposing on $x_h(t)$ and $\tilde{c}(t)$ the initial conditions $x_h(0) = 1$ and $\tilde{c}(0) = x_0$. Use this approach to construct a solution to the differential equation (15) of the form $x(t) = \tilde{c}(t)x_h(t)$. Does the result agree with the result as obtained in (b)? This example illustrates the general fact that the same initial condition can be implemented in more than one way.

Homework Problem 3: Inhomogeneous linear differential equation of third order [3]
Points: (a)[1](E); (b)[2](M); (c)[2](M,Bonus).

Consider the following third order inhomogeneous linear differential equation:

Differential equation: $$\dddot{x} - 6\ddot{x} + 11\dot{x} - 6x = f_A(t), \tag{16}$$

Initial value: $$x(0) = 1, \quad \dot{x}(0) = 0, \quad \ddot{x}(0) = a\,, \qquad \text{with} \quad a \in \mathbb{R}. \tag{17}$$

Driving: $$f_A(t) = \begin{cases} \mathrm{e}^{-bt} & \text{for} \quad t \geq 0\,, \\ 0 & \text{for} \quad t < 0\,, \end{cases} \qquad \text{with} \quad 0 < b \in \mathbb{R}. \tag{18}$$

For $t > 0$, find a general solution to this equation of the form $x(t) = x_h(t) + x_p(t)$, where $x_h(t)$ and $x_p(t)$ are the homogeneous and particular solutions to the homogeneous and inhomogeneous differential equation that have the initial values (17) or $x_p(0) = \dot{x}_p(0) = \ddot{x}_p(0) = 0$ respectively. Proceed as follows:

(a) Write the differential equation (16) in the matrix form

$$\dot{\mathbf{x}} = A \cdot \mathbf{x} + \mathbf{b}(t), \quad \text{with} \;\; \mathbf{x} \equiv (x, \dot{x}, \ddot{x})^T \equiv (x^1, x^2, x^3)^T, \;\; \mathbf{x}_0 = (x(0), \dot{x}(0), \ddot{x}(0))^T \,. \tag{19}$$

(b) Find the homogeneous solution $\mathbf{x}_h(t)$ of $(19)|_{\mathbf{b}(t)=0}$ with $\mathbf{x}_h(0) = \mathbf{x}_0$; then $x_h(t) = x^1{}_h(t)$. [Check your result: $x_h(\ln 2) = 2 + a$.]

(c) Find the inhomogeneous solution $\mathbf{x}_p(t)$ of (19), with $\mathbf{x}_p(0) = \mathbf{0}$; then $x_p(t) = x^1{}_p(t)$. [Check your result: for $a = 2$ and $b = 1$ we have $x_p(\ln 2) = \frac{7}{48}$.]

Hint: This problem is the direct analogue of the example problem on the driven, damped harmonic oscillator. The eigenvalues λ_1, λ_2, λ_3 of A are integers, with $\lambda_1 = 1$.

Homework Problem 4: System of linear differential equations with non-diagonizable matrix: critically damped harmonic oscillator [4]

Points: (a)[1](E); (b)[2](M); (c)[1](E); (d)[2](M,Bonus).

Consider a critically damped harmonic oscillator, described by the 2nd-order DEQ

$$\ddot{x} + 2\gamma\dot{x} + \gamma^2 x = 0\,. \tag{20}$$

By introducing the variables $\mathbf{x} \equiv (x, v)^T$, with $v \equiv \dot{x}$ and $\dot{v} = \ddot{x} = -\gamma^2 x - 2\gamma v$, this equation can be transcribed into a system of two first-order DEQs:

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\gamma^2 & -2\gamma \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} . \tag{21}$$

To solve the matrix equation (21), $\dot{\mathbf{x}} = A\mathbf{x}$, we may try the ansatz $\mathbf{x}(t) = \mathbf{v}\mathrm{e}^{\lambda t}$, leading to the eigenvalue problem $\lambda\mathbf{v} = A\mathbf{v}$. For the damped harmonic oscillator, this eigenvalue problem turns out to have degenerate eigenvalues. To deal with this complication, proceed as follows:

(a) Find the degenerate eigenvalue, λ, its eigenvector, $\mathbf{v}$, and the corresponding solution, $\mathbf{x}(t)$, of Eq. (21). Verify that its first component, $x(t)$, is a solution of (20). We will call this solution $x_1(t)$ henceforth.

(b) Find a second solution, $x_2(t)$, of Eq. (20) via variation of constants, by inserting the ansatz $x_2(t) = c(t)x_1(t)$ into Eq. (20). Find a differential equation for $c(t)$ and solve this equation.

(c) Using a linear combination of $x_1(t)$ and $x_2(t)$, find the solution $x(t)$ satisfying $x(0) = 1$, $\dot{x}(1) = 1$. [Check your result: if $\gamma = 2$, then $x(\ln 2) = \frac{1}{4}\left(1 - \ln 2(2 + \mathrm{e}^2)\right)$.]

(d) The critically damped harmonic oscillator can be thought of as the limit $\gamma \to \Omega$ of both the over-damped (see example problem) and under-damped (see lecture notes) harmonic oscillator. Their general solution has the form $x(t) = c_+\mathrm{e}^{\gamma_+ t} + c_-\mathrm{e}^{\gamma_- t}$, where $\gamma_\pm = -\gamma \pm \sqrt{\gamma^2 - \Omega^2}$ in the over-damped case and $\gamma_\pm = -\gamma \pm \mathrm{i}\sqrt{\Omega^2 - \gamma^2}$ in the under-damped case.

For both cases, show that a Taylor expansion of the general solution for small values of ϵt, with $\epsilon \equiv \sqrt{|\gamma^2 - \Omega^2|}$, yields expressions which can be written as linear combinations of the solutions to the critically damped harmonic oscillator found in (a) and (b).

Homework Problem 5: Series expansion for iteratively solving an equation [3]
Points: (a)[1.5](M); (b)[1.5](M).

Solve the equation $\ln[(x+1)^2] + e^y = 1 - y$ for y, to second order in the small parameter x, using the ansatz $y(x) = y_0 + y_1 x + \frac{1}{2!} y_2 x^2 + \mathcal{O}(x^3)$. Use both the methods described in the corresponding example problem:

(a) method 1: expansion of equation; and (b) method 2: repeated differentiation.

Which one do you find more convenient? [Check your results: $y_2 = \frac{1}{2}$.]

Homework Problem 6: Taylor expansion in two dimensions [2]
Points: (a)[0.5](E); (b)[1.5](M)

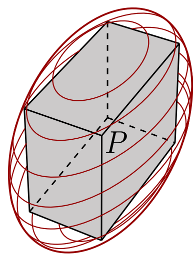
For the following functions, calculate the Taylor expansion in x and y around the point $(x, y) = (0, 0)$, up to and including second order:

(a) $f(x, y) = e^{-(x+y)^2}$, (b) $g(x, y) = \dfrac{1+x}{\sqrt{1+xy}}$.

[Check your results: the mixed second-order term in each case is: (a) $-2xy$, (b) $-\frac{1}{2}xy$.]

Homework Problem 7: Maximal volume of box enclosed in ellipsoid [2]
Points: (a)[1](M); (b)[1](M)

Consider the ellipsoid defined by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Also consider a rectangular box whose corners lie on the surface of the ellipsoid and whose edges are parallel to the elipsoid's symmetry axes. Let $P = (x_p,\, y_p,\, z_p)^T$ denote that corner of the box that lies in the positive quadrant ($x_p > 0,\, y_p > 0,\, z_p > 0$). How should this corner be chosen to maximize the volume of the box? What is the value of the maximal volume?



Hint: Maximize the volume $V(x, y, z) = 8\,xyz$ of a box having a corner at $(x, y, z)^T$, under the constraint that this point lies on the ellipsoid.
[Check your result: if $a = \frac{1}{2}$, $b = 3$, $c = \sqrt{3}$, then $V_{\max} = 4$.]

[Total Points for Homework Problems: 23]
