Sheet 09: Taylor Series. Differential Equations I

Posted: Mo 13.12.21 Central Tutorial: Fr(!) 17.12.21 Due: Th 23.12.21, 14:00
(b)[2](E/M/A) means: problem (b) counts 2 points and is easy/medium hard/advanced
Suggestions for central tutorial: example problems 2, 3, 5.
Videos exist for example problems 4 (L8.3.1).

Example Problem 1: Addition theorems for sine and cosine [1]
Points: (a)[0.5](E); (b)[0.5](E).

Prove the addition theorems for sine and cosine, for any $a, b \in \mathbb{C}$:

(a) $\cos(a+b) = \cos a \cos b - \sin a \sin b$, (b) $\sin(a+b) = \cos a \sin b + \sin a \cos b$.

Hint: Use the Euler formula on both sides of $\mathrm{e}^{\mathrm{i}(a+b)} = \mathrm{e}^{\mathrm{i}a}\mathrm{e}^{\mathrm{i}b}$.

Example Problem 2: Taylor expansions [2]
Points: (a)[1](E); (b)[1](E); (c)[1](M,Bonus).

Taylor expand the following functions. You may choose to either calculate the coefficients of the Taylor series by taking the corresponding derivatives, or to use the known Taylor expansions of $\sin(x)$, $\cos(x)$, $\frac{1}{1-x}$ and $\ln(1+x)$.

(a) $f(x) = \frac{1}{1-\sin(x)}$ around $x = 0$, up to and including fourth order.

(b) $g(x) = \sin(\ln(x))$ around $x = 1$, up to and including second order.

(c) $h(x) = \mathrm{e}^{\cos x}$ around $x = 0$, up to and including second order.

[Check your results: the highest-order term requested in each case is: (a) $\frac{2}{3}x^4$, (b) $-\frac{1}{2}(x-1)^2$, (c) $-\frac{1}{2}\mathrm{e}x^2$.]

Example Problem 3: Functions of matrices [4]
Points: (a)[0.5](E); (b)[1](E); (c)[1](M); (d)[1.5](M).

The purpose of this problem is to gain familiarity with the concept of a 'function of a matrix'. Let f be an analytic function, with Taylor series $f(x) = \sum_{l=0}^{\infty} c_l x^l$, and $A \in \mathrm{mat}(\mathbb{R}, n, n)$ a square matrix, then $f(A)$ is defined as $f(A) = \sum_{l=0}^{\infty} c_l A^l$, with $A^0 = \mathbb{1}$.

(a) A matrix A is called 'nilpotent' if an $l \in \mathbb{N}$ exists such that $A^l = 0$. Then the Taylor series of $f(A)$ ends after l terms. Example with $n = 2$: Compute e^A for $A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$.

(b) If $A^2 \propto \mathbb{1}$, then $A^{2m} \propto \mathbb{1}$ and $A^{2m+1} \propto A$, and the Taylor series for $f(A)$ has the form $f_0\mathbb{1} + f_1 A$. Example with $n = 2$: Compute e^A explicitly for $A = \theta\tilde{\sigma}$, with $\tilde{\sigma} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
[Check your result: if $\theta = -\frac{\pi}{6}$, then $\mathrm{e}^A = \frac{1}{2}\begin{pmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{pmatrix}$.]

(c) If A is diagonalizable, then $f(A)$ can be expressed in terms of its eigenvalues. Let T be the similarity transformation that diagonalizes A, with diagonal matrix $D = T^{-1}AT$ and diagonal elements $D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Show that the following relations then hold:

$$f(A) = Tf(D)T^{-1} = T \begin{pmatrix} f(\lambda_1) & 0 & \cdots & 0 \\ 0 & f(\lambda_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & f(\lambda_n) \end{pmatrix} T^{-1} .$$

Remark: Both equalities are to be established independently of each other.

(d) Now compute the matrix function e^A from (b) using diagonalization, as in (c).

Example Problem 4: Exponential representation of 2-dimensional rotation matrix [1]
Points: (a)[0.5](E); (b)[0.5](E).

The matrix $R_\theta = \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$ describes a rotation by the angle θ in $\mathbb{R}^2$. Use the following 'infinite product decomposition' to find an exponential representation of this matrix:

(a) A rotation by the angle θ can be represented as a sequence of m rotations, each by the angle θ/m: $R_\theta = [R_{(\theta/m)}]^m$. For $m \to \infty$ we have $\theta/m \to 0$, thus the matrix $R_{(\theta/m)}$ can be written as $R_{(\theta/m)} = \mathbb{1} + (\theta/m)\tilde{\sigma} + \mathcal{O}\big((\theta/m)^2\big)$ Find the matrix $\tilde{\sigma}$.

(b) Now use the identity $\lim_{m\to\infty}[1 + x/m]^m = \mathrm{e}^x$ to show that $R_\theta = \mathrm{e}^{\theta\tilde{\sigma}}$.

Remark: Justification for this identity: We have $\mathrm{e}^x = [\mathrm{e}^{x/m}]^m = [1 + x/m + \mathcal{O}\big((x/m)^2\big)]^m$. In the limit $m \to \infty$ the terms of order $\mathcal{O}\big((x/m)^2\big)$ can be neglected.

[Check your result: does the Taylor series for $\mathrm{e}^{\theta\tilde{\sigma}}$ reproduce the matrix for R_θ given above?]

Remark: The procedure illustrated here, by which an infinite sequence of identical, infinitesimal transformations is exponentiated, is a cornerstone of the theory of 'Lie groups', whose elements are associated with continuous parameters (here the angle θ). In that context the Hermitian matrix $\mathrm{i}\tilde{\sigma}$ is called the 'generator' of the rotation.

Example Problem 5: Separation of variables [2]
Points: (a)[1](E); (b)[1](E).

A first-order differential equation is called **autonomous** if it has the form $\dot{x} = f(x)$, i.e. the right hand side is time independent (non-autonomous equations have $\dot{x} = f(x,t)$). Such an equation can the solved by separation of variables.

(a) Consider the autonomous differential equation $\dot{x} = x^2$ for the function $x(t)$. Solve it by separation of variables for two different initial conditions: (i) $x(0) = 1$ and (ii) $x(2) = -1$. [Check your results: (i) $x(-2) = \frac{1}{3}$, and (ii) $x(2) = -1$.]

(b) Sketch your solutions qualitatively. Convince yourself that your sketches for the function $x(t)$ and its derivative $\dot{x}(t)$ satisfy the relation specified by the differential equation.

Example Problem 6: Separation of variables: barometric formula [1]
Points: [1](E).

The standard barometric formula for atmospheric pressure, $p(x)$, as a function of the height, x, is given by: $\frac{\mathrm{d}p(x)}{\mathrm{d}x} = -\alpha \frac{p(x)}{T(x)}$. Solve this equation with initial value $p(x_0) = p_0$ for the case of a linear temperature gradient, $T(x) = T_0 - b(x - x_0)$.
[Check your result: if $\alpha, b, T_0, x_0, p_0 = 1$, then $p(1) = 1$.]

Example Problem 7: Linear homogeneous differential equation with constant coefficients [2]
Points: [2](E).

Use an exponential ansatz to solve the following differential equation:

$$\dot{\mathbf{x}}(t) = A \cdot \mathbf{x}(t) , \qquad A = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ -4 & -3 \end{pmatrix} , \qquad \mathbf{x}(0) = (2,1)^T .$$

[Total Points for Example Problems: 13]

Homework Problem 1: Powers of Sine and Cosine [1]
Points: (a)[0.5](E); (b)[0.5](E)

Use the Euler-de Moivre identity to prove the following identities, for any $a \in \mathbb{C}$:

(a) $\cos^2 a = \frac{1}{2} + \frac{1}{2}\cos(2a)\,,$ $\qquad \sin^2 a = \frac{1}{2} - \frac{1}{2}\cos(2a)\,.$

(b) $\cos^3 a = \frac{3}{4}\cos a + \frac{1}{4}\cos(3a)\,,$ $\qquad \sin^3 a = \frac{3}{4}\sin a - \frac{1}{4}\sin(3a)\,.$

Homework Problem 2: Taylor expansions [3]
Points: (a)[1](E); (b)[1](E); (c)[1](M)

Taylor expand the following functions. You may choose to either calculate the coefficients of the Taylor series by taking the corresponding derivatives, or to use the known Taylor expansions of $\sin(x)$, $\cos(x)$, $\frac{1}{1-x}$ and $\ln(1+x)$.

(a) $f(x) = \frac{\cos(x)}{1-x}$ around $x = 0$. Keep all terms up to and including third order.

(b) $g(x) = \mathrm{e}^{\cos(x^2+x)}$ about $x = 0$, up to and including third order.

(c) $h(x) = \mathrm{e}^{-x}\ln(x)$ around $x = 1$, up to and including third order.

[Check your results: the highest-order term requested in each case is: (a) $\frac{1}{2}x^3$, (b) $-\mathrm{e}\,x^3$, (c) $\frac{4}{3}\mathrm{e}^{-1}(x-1)^3$.]

Homework Problem 3: Functions of matrices [3]
Points: (a)[0.5](E); (b)[1](E); (c)[1.5](M); (d)[1](A,Bonus).

Express each of the following matrix functions explicitly in terms of a matrix:

(a) e^A, with $A = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$.

(b) e^B, with $B = b\sigma_1$ and $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, using the Taylor series of the exponential function.

[Check your result: if $b = \ln 2$, then $\mathrm{e}^B = \frac{1}{4}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$.]

(c) The same function as in (b), now by diagonalizing B.

(d) e^C, with $C = \mathrm{i}\theta\,\Omega$, where $\Omega = n_j S_j$, while $\mathbf{n} = (n_1, n_2, n_3)^T$ is a unit vector ($\|\mathbf{n}\| = 1$) and S_j are the spin-$\frac{1}{2}$ matrices: $S_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $S_2 = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$, $S_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
Hint: Start by computing Ω^2 (for this, the property $S_i S_j + S_j S_i = \frac{1}{2}\delta_{ij}\mathbb{1}$ of the spin-$\frac{1}{2}$ matrices is useful), and then use the Taylor series of the exponential function.
$\big[$Check your result: if $\theta = -\frac{\pi}{2}$ and $n_1 = -n_2 = n_3 = \frac{1}{\sqrt{3}}$, then $\mathrm{e}^C = \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{3}-\mathrm{i} & 1-\mathrm{i} \\ -1-\mathrm{i} & \sqrt{3}+\mathrm{i} \end{pmatrix}$.$\big]$

Remark: The exponential form e^C is a representation of $\mathrm{SU}(2)$ transformations, the group of all special unitary transformations in $\mathbb{C}^2$. Its elements are characterized by three continuous real parameters (here θ, n_1 and n_2, with $n_3 = \sqrt{1 - n_1^2 - n_2^2}$). The S_j matrices are 'generators' of these transformations; they satisfy the $\mathrm{SU}(2)$ algebra, i.e. their commutators yield $[S_i, S_j] = \mathrm{i}\epsilon_{ijk} S_k$.

Homework Problem 4: Exponential representation 3-dimensional rotation matrix [4]
Points: (a)[1](E); (b)[1](M); (c)[1](M); (d)[1](A)

In $\mathbb{R}^3$, a rotation by an angle θ, about an axis whose direction is given by the unit vector $\mathbf{n} = (n_1, n_2, n_3)$, is represented by a 3×3 matrix that has the following matrix elements:

$$(R_\theta(\mathbf{n}))_{ij} = \delta_{ij}\cos\theta + n_i n_j (1 - \cos\theta) - \epsilon_{ijk}\, n_k \sin\theta \qquad (\epsilon_{ijk} = \text{Levi-Civita symbol})\,. \tag{1}$$

The goal of the following steps is to supply a justification for Eq. (1).

(a) Consider first the three matrices $R_\theta(\mathbf{e}_i)$ for rotations by the angle θ about the three coordinate axes $\mathbf{e}_i$, with $i = 1, 2, 3$. Elementary geometrical considerations yield:

$$R_\theta(\mathbf{e}_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \quad R_\theta(\mathbf{e}_2) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}, \quad R_\theta(\mathbf{e}_3) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For each of these matrices, use an infinite product decomposition of the form $R_\theta(\mathbf{n}) = \lim_{m\to\infty}[R_{\theta/m}(\mathbf{n})]^m$ to obtain an exponential representation of the form $R_\theta(\mathbf{e}_i) = \mathrm{e}^{\theta\tau_i}$. Find the three 3×3 matrices τ_1, τ_2 and τ_3. [Check your results: The τ_i commutators yield $[\tau_i, \tau_j] = \epsilon_{ijk}\tau_k$. This is the so-called $\mathrm{SO}(3)$ algebra, which underlies the representation theory of 3-dimensional rotations. Moreover, $\tau_1^2 + \tau_2^2 + \tau_3^2 = -2\mathbb{1}$.]

(b) Now consider a rotation by the angle θ about an arbitrary axis $\mathbf{n}$. To find an exponential representation for it using an infinite product decomposition, we need an approximation for $R_{\theta/m}(\mathbf{n})$ up to first order in the small angle θ/m. It has the following form:

$$R_{\theta/m}(\mathbf{n}) = R_{n_1\theta/m}(\mathbf{e}_1) R_{n_2\theta/m}(\mathbf{e}_2) R_{n_3\theta/m}(\mathbf{e}_3) + \mathcal{O}\big((\theta/m)^2\big). \tag{2}$$

Intuitive justification: If the rotation angle θ/m is sufficiently small, the rotation can be performed in three substeps, each about a different direction $\mathbf{e}_i$, by the 'partial' angle $n_i\theta/m$. The prefactors n_i ensure that for $\mathbf{n} = \mathbf{e}_i$ (rotation about a coordinate axis i) only *one* of the three factors in (2) is different from $\mathbb{1}$, namely the one that yields $R_{\theta/m}(\mathbf{e}_i)$; for example, for $\mathbf{n} = \mathbf{e}_2 = (0, 1, 0)^T$: $R_{0\theta/m}(\mathbf{e}_1) R_{1n_2\theta/m}(\mathbf{e}_2) R_{0\theta/m}(\mathbf{e}_3) = R_{n_2\theta/m}(\mathbf{e}_2)$.

Show that such a product decomposition of $R_\theta(\mathbf{n})$ yields the following exponential representation:

$$R_\theta(\mathbf{n}) = \mathrm{e}^{\theta\Omega}\,, \qquad \Omega = n_i\tau_i = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}\,, \qquad (\Omega)_{ij} = -\epsilon_{ijk}n_k\,. \tag{3}$$

(c) Show that Ω, the 'generator' of the rotation, has the following properties:

$$(\Omega^2)_{ij} = n_i n_j - \delta_{ij}, \qquad \Omega^l = -\Omega^{l-2} \quad \text{for } 3 \le l \in \mathbb{N}. \qquad \text{[Cayley-Hamilton theorem]} \tag{4}$$

Hint: First compute Ω^2 and Ω^3, then the form of $\Omega^{l>3}$ will be obvious.

(d) Show that the Taylor expansion of $R_\theta(\mathbf{n}) = \mathrm{e}^{\theta\Omega}$ yields the following expression,

$$R_\theta(\mathbf{n}) = \mathbb{1} + \Omega\sin\theta + \Omega^2(1-\cos\theta), \tag{5}$$

and that its matrix elements correspond to Eq. (1).

Homework Problem 5: Separation of variables [2]
Points: (a)[1](E); (b)[1](E)

(a) Consider the differential equation $y' = -x^2/y^3$ for the function $y(x)$. Solve it by separation of variables, for two different initial conditions: (i) $y(0) = 1$, and (ii) $y(0) = -1$. [Check your result: (i) $y(-1) = \left(\frac{7}{3}\right)^{1/4}$, (ii) $y(-1) = -\left(\frac{7}{3}\right)^{1/4}$.]

(b) Sketch your solutions qualitatively. Convince yourself that your sketches for the function $y(x)$ and its derivative $y'(x)$ satisfy the relation specified by the differential equation.

Homework Problem 6: Separation of variables: bacterial culture with toxin [4]
Points: (a)[1](E); (b)[1](M); (c)[1](E); (d)[1](E)

A bacterial culture is exposed to the effects of a toxin. The death rate induced by the toxin is proportional to the number, $n(t)$, of bacteria still alive in the culture at a time t and the amount of toxin, $T(t)$, remaining in the system, which is given by $\tau n(t)T(t)$, where τ is a positive constant. On the other hand, the natural growth rate of the bacteria in the culture is exponential, i.e. it grows with a rate $\gamma n(t)$, with $\gamma > 0$. In total, the number of bacteria in the culture is given by the differential equation

$$\dot{n} = \gamma n - \tau n T(t), \quad \text{for } t \ge 0.$$

(a) Find the general solution to this linear DEQ, with initial condition $n(0) = n_0$.

(b) Assume now that the toxin is injected into the system at a constant rate $T(t) = at$, where $a > 0$. Use a qualitative analysis of the differential equation (i.e. without solving it explicitly) to show that the bacterial population grows up to a time $t = \gamma/(a\tau)$, and decreases thereafter. Furthermore, show that as $t \to \infty$, $n(t) \to 0$, i.e. the bacterial culture is practically wiped out.

(c) Now find the explicit solution, $n(t)$, to the differential equation and sketch $n(t)$ qualitatively as a function of t. Convince yourself that the sketch fulfils the relation between $n(t)$, $\dot{n}(t)$ and t that is specified by the differential equation. [Check your result: if $\tau = 1$, $a = 1$, $n_0 = 1$ and $\gamma = \sqrt{\ln 2}$, then $n(\sqrt{\ln 2}) = \sqrt{2}$.]

(d) Find the time t_h at which the number of bacteria in the culture drops to half the initial value. [Check your result: if $\tau = 4$, $a = 2/\ln 2$ and $\gamma = 3$, then $t_h = \ln 2$.]

Homework Problem 7: Linear homogeneous differential equation with constant coefficients [2]
Points: [2](E).

Use an exponential ansatz to solve the following differential equation:

$$\dot{\mathbf{x}}(t) = A \cdot \mathbf{x}(t) \ , \qquad A = \frac{1}{2}\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \ , \qquad \mathbf{x}(0) = (1,3)^T \ .$$

[Total Points for Homework Problems: 19]
