Sheet 03: Vector Product, Curves, Line Integrals

Posted: Mo 01.11.21 Central Tutorial: Th 04.11.21 Due: Th 11.11.21, 14:00
(b)[2](E/M/A) means: problem (b) counts 2 points and is easy/medium hard/advanced
Suggestions for central tutorial: example problems 3, 6, 7, 4.
Videos exist for example problems 4 (L4.3.1), 8 (V1.4.1).

Example Problem 1: $1/(1-x^2)$ Integrals by hyperbolic substitution [3]
Points: (a)[1](E); (b)[2](M)

For integrals involving $1/(1-x^2)$, the substitution $x = \tanh y$ may help, since it gives $1 - x^2 = \mathrm{sech}^2\, y$. Use it to compute the following integrals $I(z)$; check your answers by calculating $\frac{\mathrm{d}I(z)}{\mathrm{d}z}$. $\big[$Check your results: (a) $I\big(\frac{3}{5}\big) = \ln 2$; (b) for $a = 3$, $I\big(\frac{1}{5}\big) = \frac{1}{6}\ln 2 + \frac{5}{32}$.$\big]$

(a) $I(z) = \displaystyle\int_0^z \mathrm{d}x\, \frac{1}{1-x^2} \quad (|z| < 1)$, (b) $I(z) = \displaystyle\int_0^z \mathrm{d}x\, \frac{1}{(1-a^2x^2)^2} \quad (|az| < 1)$.

Hint: For (b), use integration by parts for the $\cosh^2 y$ integral emerging after the substitution.

Example Problem 2: Elementary computations with vectors [3]
Points: (a)[1](E); (b)[1](E); (c)[1](E)

Given the vectors $\mathbf{a} = (4,3,1)^T$ and $\mathbf{b} = (1,-1,1)^T$.

(a) Calculate $\|\mathbf{b}\|$, $\mathbf{a} - \mathbf{b}$, $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$.

(b) Decompose $\mathbf{a} \equiv \mathbf{a}_\parallel + \mathbf{a}_\perp$ into two vectors parallel and perpendicular to $\mathbf{b}$.

(c) Calculate $\mathbf{a}_\parallel \cdot \mathbf{b}$, $\mathbf{a}_\perp \cdot \mathbf{b}$, $\mathbf{a}_\parallel \times \mathbf{b}$ and $\mathbf{a}_\perp \times \mathbf{b}$. Do these results match your expectations?

[Check your results: (a) $\mathbf{a} \cdot \mathbf{b} + \sum_i (\mathbf{a} \times \mathbf{b})^i = -4$, (b) $\sum_i (\mathbf{a}_\parallel)^i = \frac{2}{3}$, $\sum_i (\mathbf{a}_\perp)^i = 7\frac{1}{3}$.]

Example Problem 3: Levi-Civita symbol [3]
Points: (a)[1](E); (b)[1](E); (c)[1](E).

(a) Is the statement $a^i b^j \epsilon_{ij2} \stackrel{?}{=} -a^k \epsilon_{k2l} b^l$ true or false? Justify your answer.

Express the following k-sums over products of two Levi-Civita symbols in terms of Kronecker delta symbols. Check your answers by also writing out the k-sums explicitly and evaluating each term separately.

(b) $\epsilon_{1ik}\epsilon_{kj1}$, (c) $\epsilon_{1ik}\epsilon_{kj2}$.

Example Problem 4: Grassmann identity (BAC-CAB) and Jacobi identity [5]
Points: (a)[2](M); (b)[1](E); (c)[2](M)

(a) Prove the Grassmann (or BAC-CAB) identity for arbitrary vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in \mathbb{R}^3$:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}).$$

Hint: Expand the three vectors in an orthonormal basis, e.g. $\mathbf{a} = \mathbf{e}_i a^i$, and use the identity $\epsilon_{ijk}\epsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$ for the Levi-Civita symbol. If you prefer, you may equally well write all indices downstairs, e.g. $\mathbf{a} = \mathbf{e}_i a_i$, since in an orthonormal basis $a_i = a^i$.

(b) Use the Grassmann identity to derive the Jacobi identity:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}\,.$$

(c) Check both identities explicitly for $\mathbf{a} = (1,1,2)^T$, $\mathbf{b} = (3,2,0)^T$ and $\mathbf{c} = (2,1,1)^T$ by separately computing all terms they contain.

Example Problem 5: Scalar triple product [2]
Points: (a)[0.5](E); (b)[1](E); (c)[0.5](E)

This problem illustrates an important relation between the scalar triple product and the question whether three vectors in $\mathbb{R}^3$ are linearly independent or not.

(a) Compute the scalar triple product, $S(y) = \mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)$, of $\mathbf{v}_1 = (1,0,2)^T$, $\mathbf{v}_2 = (3,2,1)^T$ and $\mathbf{v}_3 = (-1,-2,y)^T$ as a function of the variable y. $\big[$Check your result: $S(1) = -4\,\big]$.

(b) By solving the vector equation $\mathbf{v}_i a^i = \mathbf{0}$, find that value of y for which $\mathbf{v}_1$, $\mathbf{v}_2$ are $\mathbf{v}_3$ *not* linearly independent.

(c) What is the value of $S(y)$ for the value of y found in (b)? Interpret your result!

Example Problem 6: Velocity and acceleration [3]
Points: (a)[1](E); (b)[1](M); (c)[1](E)

Consider the curve $\gamma = \{\mathbf{r}(t)\,|\,t \in (0, 2\pi/\omega)\}$, $\mathbf{r}(t) = (aC(t), S(t))^T \in \mathbb{R}^2$, with $C(t) = \cos[\pi(1-\cos\omega t)]$, $S(t) = \sin[\pi(1-\cos\omega t)]$, and $0 < a, \omega \in \mathbb{R}$.

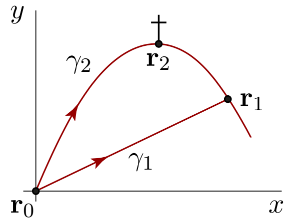
(a) Calculate the curve's velocity vector, $\dot{\mathbf{r}}(t)$, and it's acceleration vector, $\ddot{\mathbf{r}}(t)$. Can $\mathbf{r}(t)$ be expressed in terms of $\dot{\mathbf{r}}(t)$ and $\ddot{\mathbf{r}}(t)$?

(b) Can you represent the curve without the parameter t using an equation? Do you recognize the curve? Sketch the curve for the case $a = 2$.

(c) Calculate $\mathbf{r}(t) \cdot \dot{\mathbf{r}}(t)$. For which values of a is $\mathbf{r}(t) \cdot \dot{\mathbf{r}}(t) = 0$ true for all t?

Example Problem 7: Line integral: mountain hike [3]
Points: [3](M)

Two hikers want to hike from the point $\mathbf{r}_0 = (0,\,0)^T$ in the valley to a mountain hut at the point $\mathbf{r}_1 = (3,\,3a)^T$. Hiker 1 chooses the straight path from valley to hut, γ_1. Hiker 2 chooses a parabolic path, γ_2, via the mountain top at the apex of the parabola, at $\mathbf{r}_2 = (2,\,4a)^T$ (see figure). They are acted on by the force of gravity $\mathbf{F}_g = -10\,\mathbf{e}_y$, and a height-dependent wind force, $\mathbf{F}_w = -y^2\,\mathbf{e}_x$.



Find the work, $W[\gamma_i] = -\int_{\gamma_i} \mathrm{d}\mathbf{r}\cdot\mathbf{F}$, performed by the hikers along γ_1 and γ_2, as function of the parameter a. $\big[$Check your results: for $a=1$ one finds $W[\gamma_1]=39$, $W[\gamma_2]=303/5.\big]$

[Total Points for Example Problems: 22]

Homework Problem 1: $1/(1+x^2)$ Integrals by trigonometric substitution [3]
Points: (a)[1](E); (b)[2](M).

For integrals involving $1/(1+x^2)$, the substitution $x=\tan y$ may help, since it gives $1+x^2=\sec^2 y$. Use it to compute the following integrals $I(z)$; check your answers by calculating $\frac{\mathrm{d}I(z)}{\mathrm{d}z}$.
$\big[$Check your results: (a) $I(1)=\frac{\pi}{4}$; (b) for $a=\frac{1}{2}$, $I(2)=\frac{\pi}{4}+\frac{1}{2}.\big]$

(a) $I(z) = \displaystyle\int_0^z \mathrm{d}x\,\frac{1}{1+x^2}$ (b) $I(z) = \displaystyle\int_0^z \mathrm{d}x\,\frac{1}{(1+a^2x^2)^2}$.

Homework Problem 2: Elementary computations with vectors [3]
Points: (a)[1](E); (b)[1](E); (c)[1](E)

Given the vectors $\mathbf{a}=(2,1,5)^T$ and $\mathbf{b}=(-4,3,0)^T$.

(a) Calculate $\|\mathbf{b}\|$, $\mathbf{a}-\mathbf{b}$, $\mathbf{a}\cdot\mathbf{b}$ and $\mathbf{a}\times\mathbf{b}$.

(b) Decompose $\mathbf{a}$ into a vector $\mathbf{a}_\parallel$ parallel to $\mathbf{b}$ and a vector $\mathbf{a}_\perp$ perpendicular to $\mathbf{b}$.

(c) Calculate $\mathbf{a}_\parallel\cdot\mathbf{b}$, $\mathbf{a}_\perp\cdot\mathbf{b}$, $\mathbf{a}_\parallel\times\mathbf{b}$ and $\mathbf{a}_\perp\times\mathbf{b}$. Do these results match your expectations?

[Check your results: (a) $\mathbf{a}\cdot\mathbf{b}+\sum_i(\mathbf{a}\times\mathbf{b})^i=-30$, (b) $\sum_i(\mathbf{a}_\parallel)^i=\frac{1}{5}$, $\sum_i(\mathbf{a}_\perp)^i=7\frac{4}{5}$.]

Homework Problem 3: Levi-Civita symbol [2]
Points: (a)[0,5](E); (b)[0,5](E); (c)[0,5](E); (d)[0,5](E).

(a) Is the statement $a^i a^j \epsilon_{ij3} \stackrel{?}{=} b^m b^n \epsilon_{mn2}$ true or false? Justify your answer.

Express the following k-sums over products of two Levi-Civita symbols in terms of Kronecker delta functions.

(b) $\epsilon_{1ik}\epsilon_{23k}$, (c) $\epsilon_{2jk}\epsilon_{ki2}$, (d) $\epsilon_{1ik}\epsilon_{k3j}$.

Homework Problem 4: Lagrange identity [3]
Points: (a)[1](E); (b)[1](E); (c)[1](E)

(a) Prove the Lagrange identity for arbitrary vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d} \in \mathbb{R}^3$:

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}).$$

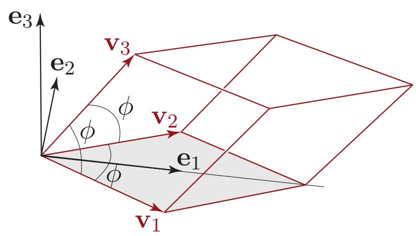
Hint: Work in an orthonormal basis and use the properties of the Levi-Civita symbol.

(b) Use (a) to compute $\|\mathbf{a} \times \mathbf{b}\|$ and express the result in terms of $\|\mathbf{a}\|$, $\|\mathbf{b}\|$ and the angle ϕ between $\mathbf{a}$ and $\mathbf{b}$.

(c) Check the Lagrange identity explicitly for the vectors $\mathbf{a} = (2, 1, 0)^T$, $\mathbf{b} = (3, -1, 2)^T$, $\mathbf{c} = (3, 0, 2)^T$, $\mathbf{d} = (1, 3, -2)^T$, by separately computing all its terms.

Homework Problem 5: Scalar triple product [3]
Points: [3](M)

Compute the volume, $V(\phi)$, of the parallelepiped spanned by three unit vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$, each pair of which encloses a mutual angle of ϕ (with $0 \leq \phi \leq \frac{2}{3}\pi$; why is this restriction needed?).
Check your results: (i) What do you expect for $V(\frac{\pi}{2})$ and $V(\frac{2}{3}\pi)$? (ii): $V(\frac{\pi}{3}) = \frac{1}{\sqrt{2}}$.



Hint: Choose the orientation of the parallelepiped such that $\mathbf{v}_1$ and $\mathbf{v}_2$ both lie in the plane spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$, and that $\mathbf{e}_1$ bisects the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$ (see figure).

Homework Problem 6: Velocity and acceleration [2]
Points: (a)[1](E); (b)[0.5](E); (c)[0.5](E)

Consider the curve $\gamma = \{\mathbf{r}(t) \,|\, t \in (0, \infty)\}$, $\mathbf{r}(t) = (\mathrm{e}^{-t^2}, a\mathrm{e}^{t^2})^T \in \mathbb{R}^2$, with $0 < a \in \mathbb{R}$ ($0 < a < 1$ for (c)).

(a) Calculate the curve's velocity vector, $\dot{\mathbf{r}}(t)$, and it's acceleration vector, $\ddot{\mathbf{r}}(t)$. Can $\mathbf{r}(t)$ be expressed as a linear combination of $\dot{\mathbf{r}}(t)$ and $\ddot{\mathbf{r}}(t)$?

(b) Can you represent the curve without the parameter t using an equation? Do you recognize the curve? Sketch the curve for the case $a = 2$.

(c) Calculate $\mathbf{r}(t) \cdot \dot{\mathbf{r}}(t)$. Find the time, $t(a)$, for which $\mathbf{r}(t) \cdot \dot{\mathbf{r}}(t) = 0$ holds. [Check your result: $t(\mathrm{e}^{-2}) = \pm 1$.]

Homework Problem 7: Line integrals in Cartesian coordinates [4]
Points: (a)[2](M); (b)[1](E); (c)[1](M)

Let $\mathbf{F}(\mathbf{r}) = (x^2, z, y)^T$ be a three-dimensional vector field in Cartesian coordinates, with $\mathbf{r} = (x, y, z)^T$. Calculate the line integral $\int_\gamma \mathrm{d}\mathbf{r} \cdot \mathbf{F}$ along the following paths from $\mathbf{r}_0 \equiv (0, 0, 0)^T$ to $\mathbf{r}_1 \equiv (0, -2, 1)^T$:

(a) $\gamma_a = \gamma_1 \cup \gamma_2$ is the composite path consisting of γ_1, the straight line from $\mathbf{r}_0$ to $\mathbf{r}_2 \equiv (1, 1, 1)^T$, and γ_2, the straight line from $\mathbf{r}_2$ to $\mathbf{r}_1$.

(b) γ_b is parametrized by $\mathbf{r}(t) = (\sin(\pi t), -2t^{1/2}, t^2)^T$, with $0 < t < 1$.

(c) γ_c is a parabola in the yz-plane with the form $z(y) = y^2 + \frac{3}{2}y$.

[Check your results: the sum of the answers from (a), (b) and (c) is -6.]

[Total Points for Homework Problems: 20]
