LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

Fakultät für Physik
R: Rechenmethoden für Physiker, WiSe 2021/22
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Sheet 02: Vector Spaces, Euclidean Spaces

Posted: Mo 25.10.21 Central Tutorial: Th 28.10.21 Due: Th 04.11.21, 14:00
(b)[2](E/M/A) means: problem (b) counts 2 points and is easy/medium hard/advanced
Suggestions for central tutorial: example problems 5, 7, 9, 8.
Videos exist for example problems 4 (L2.4.1), 9 (L3.3.7).

Example Problem 1: $\sqrt{1-x^2}$ Integrals by trigonometric substitution [3]
Points: (a)[1](E); (b)[2](M)

For integrals involving $\sqrt{1-x^2}$, the substitution $x = \sin y$ may help, since it gives $\sqrt{1-x^2} = \cos y$. Use it to compute the following integrals $I(z)$; check your answers by calculating $\frac{\mathrm{d}I(z)}{\mathrm{d}z}$.
$\big[$Check your results: (a) $I\big(\frac{1}{\sqrt{2}}\big) = \frac{\pi}{4}$; (b) for $a = \frac{1}{2}$, $I\big(\sqrt{2}\big) = \frac{\pi}{4} + \frac{1}{2}$.$\big]$

(a) $I(z) = \displaystyle\int_0^z \mathrm{d}x \, \frac{1}{\sqrt{1-x^2}} \quad (|z| < 1),$ (b) $I(z) = \displaystyle\int_0^z \mathrm{d}x \, \sqrt{1-a^2x^2} \quad (|az| < 1).$

Hint: For (b), use integration by parts for the $\cos^2 y$ integral emerging after the substitution.

Example Problem 2: Vector space axioms: rational numbers [3]
Points: (a)[2,5](E); (b)[0,5](E).

(a) Show that the set $\mathbb{Q}^2 = \{\binom{x^1}{x^2} \,|\, x^1, x^2 \in \mathbb{Q}\}$, consisting of all pairs of rational numbers, forms a $\mathbb{Q}$-vector space over the field of rational numbers.

(b) Is it possible to construct a vector space from the set of all pairs of integers, $\mathbb{Z}^2 = \{\binom{x^1}{x^2} \,|\, x^1, x^2 \in \mathbb{Z}\}$? Justify your answer!

Example Problem 3: Real vector space with unconventional composition rules [Bonus]
Points: [2](M,Bonus)

The axioms that define a vector space can be satisfied in many different ways. These may involve unconventional definitions of vector addition and scalar multiplication. The purpose of the present problem is to illustrate this point.
For any $a \in \mathbb{R}$, let $V_a \equiv \{\mathbf{v}_x\}$ be a set whose elements $\mathbf{v}_x$, labelled by real numbers $x \in \mathbb{R}$, satisfy the following composition rules:

Addition:	$\boldsymbol{+}$:	$V_a \times V_a \to V_a,$	$(\mathbf{v}_x, \mathbf{v}_y) \mapsto \mathbf{v}_x \boldsymbol{+} \mathbf{v}_y \equiv \mathbf{v}_{x+y+a}$
Multiplication by a scalar:	$\cdot$:	$\mathbb{R} \times V_a \to V_a,$	$(\lambda, \mathbf{v}_x) \mapsto \lambda \boldsymbol{\cdot} \mathbf{v}_x \equiv \mathbf{v}_{\lambda x + a(\lambda-1)}$

The a and x labels, being real numbers, satisfy the usual addition and scalar multiplication rules of $\mathbb{R}$; e.g. in V_2 we have: $\mathbf{v}_3 \boldsymbol{+} \mathbf{v}_4 = \mathbf{v}_{3+4+2} = \mathbf{v}_9$ and $3 \boldsymbol{\cdot} \mathbf{v}_4 = \mathbf{v}_{3\cdot 4+2(3-1)} = \mathbf{v}_{16}$. Show that the triple $(V_a, \boldsymbol{+}, \boldsymbol{\cdot})$ represents an $\mathbb{R}$-vector space, with $\mathbf{v}_{-a}$ and 1 being the neutral elements for addition and scalar multiplication, respectively, and $\mathbf{v}_{-x-2a}$ the additive inverse of $\mathbf{v}_x$.

Example Problem 4: Linear independence [3]
Points: (a)[2](M); (b)[1](M)

(a) Are the vectors $\mathbf{v}_1 = (0, 1, 2)^T$, $\mathbf{v}_2 = (1, -1, 1)^T$ and $\mathbf{v}_3 = (2, -1, 4)^T$ linearly independent?

(b) Depending on whether your answer is yes or no, find a vector $\mathbf{v}_2'$ such that $\mathbf{v}_1$, $\mathbf{v}_2'$ and $\mathbf{v}_3$ are linearly dependent or independent, respectively, and show explicitly that they have this property.

Example Problem 5: Einstein summation convention [2]
Points: (a)[0.5](E), (b)[0.5](E), (c)[0.5](E), (d)[0.5](E).

Let $a_1, a_2, b^1, b^2 \in \mathbb{R}$. Which of the following statements, formulated using the Einstein summation convention, are true and which are false? Justify your answers!

(a) $a_i b^i \stackrel{?}{=} b^j a_j \,,$

(b) $a_i \delta^i{}_j b^j \stackrel{?}{=} a_k b^k \,,$

(c) $a_i b^j a_j b^k \stackrel{?}{=} a_k b^l a_l b^i \,,$

(d) $a_1 a_i b^1 b^i + b^2 a_j a_2 b^j \stackrel{?}{=} (a_i b^i)^2 \,.$

Example Problem 6: Angle, orthogonal decomposition [2]
Points: (a)[0,5](E); (b)[1.5](E).

(a) Find the angle between the vectors $\mathbf{a} = (3, 4)^T$ and $\mathbf{b} = (7, 1)^T$.

(b) Consider the vectors $\mathbf{c} = (3, 1)^T$ and $\mathbf{d} = (-1, 2)^T$. Decompose $\mathbf{c} = \mathbf{c}_{\parallel} + \mathbf{c}_{\perp}$ into components parallel and perpendicular to $\mathbf{d}$, respectively. Scetch all four vectors.
[Check your results: $\|\mathbf{c}_{\parallel}\| = \frac{1}{\sqrt{5}}$, $\|\mathbf{c}_{\perp}\| = \frac{7}{\sqrt{5}}$.]

Example Problem 7: Projection onto an orthonormal basis [2]
Points: (a)[1](E); (b)[1](E)

(a) Show that the vectors $\mathbf{e}_1' = \frac{1}{\sqrt{2}}(1, 1)^T$, $\mathbf{e}_2' = \frac{1}{\sqrt{2}}(1, -1)^T$ form an orthonormal basis for $\mathbb{R}^2$.

(b) Express the vector $\mathbf{w} = (-2, 3)^T$ in the form $\mathbf{w} = \mathbf{e}_1' w^1 + \mathbf{e}_2' w^2$, by computing its components w^i with respect to the basis $\{\mathbf{e}_i'\}$ through projection onto the basis vectors. [Check your results: $\sum_{i=1}^2 w^i = -2\sqrt{2}$.]

Example Problem 8: Gram-Schmidt procedure [2]
Points: [2](E)

Apply the Gram-Schmidt procedure to the following set of linearly independent vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to construct an orthonormal set $\{\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'\}$ with the same span and with $\mathbf{e}_1' || \mathbf{v}_1$.

$$\mathbf{v}_1 = (1, -2, 1)^T \,, \qquad \mathbf{v}_2 = (1, 1, 1)^T \,, \qquad \mathbf{v}_3 = (0, 1, 2)^T \,.$$

Example Problem 9: Non-orthogonal basis and metric [4]
Points: (a)[1](E); (b)[1](E); (c)[1](M); (d)[1](M)

Consider the vectors $\hat{\mathbf{v}}_1 = \binom{2}{0}$ and $\hat{\mathbf{v}}_2 = \binom{1}{1}$, written as column vectors in the standard basis of $\mathbb{R}^2$. (In this problem we use the following notation: vectors in the inner product space $\mathbb{R}^2$ carry a caret, e.g. $\hat{\mathbf{x}}$, and their components w.r.t. a given basis do not, e.g. $\mathbf{x}$.)

(a) Write the standard basis vector $\hat{\mathbf{e}}_1 = \binom{1}{0}$ as a linear combination of $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$. Ditto for $\hat{\mathbf{e}}_2 = \binom{0}{1}$. Do $\{\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2\}$ form a basis for $\mathbb{R}^2$?

(b) Let $\hat{\mathbf{x}} = \hat{\mathbf{v}}_1 x^1 + \hat{\mathbf{v}}_2 x^2$ and $\hat{\mathbf{y}} = \hat{\mathbf{v}}_1 y^1 + \hat{\mathbf{v}}_2 y^2$ be two vectors in $\mathbb{R}^2$, whose components w.r.t. $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$ are given by $\mathbf{x} = (x^1, x^2)^T = (3, -4)^T$ and $\mathbf{y} = (y^1, y^2)^T = (-1, 3)^T$ respectively. Express $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ as column vectors in the standard basis of $\mathbb{R}^2$ and compute their scalar product $\langle \hat{\mathbf{x}}, \hat{\mathbf{y}} \rangle_{\mathbb{R}^2}$.

(c) If the scalar product $\langle \hat{\mathbf{x}}, \hat{\mathbf{y}} \rangle_{\mathbb{R}^2}$ is expressed through the components x^i of $\hat{\mathbf{x}}$ and y^i of $\hat{\mathbf{y}}$ w.r.t. the non-orthogonal basis $\{\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2\}$, then it takes the form of an inner product with a metric: $\langle \hat{\mathbf{x}}, \hat{\mathbf{y}} \rangle_{\mathbb{R}^2} = \langle \mathbf{x}, \mathbf{y} \rangle_g = x^i g_{ij} y^j$, with $g_{ij} = \langle \hat{\mathbf{v}}_i, \hat{\mathbf{v}}_j \rangle_{\mathbb{R}^2}$. Compute the components of the metric explicitly (concretely: find g_{11}, g_{12}, g_{21} and g_{22}).

(d) The inner product from (c) can be written as $\langle \hat{\mathbf{x}}, \hat{\mathbf{y}} \rangle_{\mathbb{R}^2} = (x^i g_{ij}) y^j = x_j y^j$, with $x_j = x^i g_{ij}$, thus "hiding" the metric by absorbing it into the definition of covariant components (with subscript indices). Compute $\langle \hat{\mathbf{x}}, \hat{\mathbf{y}} \rangle_{\mathbb{R}^2}$ in this manner, by first finding x_1 and x_2. [Check: is the result consistent with that from (b)?]

[Total Points for Example Problems: 21]

Homework Problem 1: $\sqrt{1+x^2}$ Integrals by hyperbolic substitution [3]
Points: (a)[1](E); (b)[2](M)

For integrals involving $\sqrt{1+x^2}$, the substitution $x = \sinh y$ may help, since it gives $\sqrt{1+x^2} = \cosh y$. Use it to compute the following integrals $I(z)$; check your answers by calculating $\frac{\mathrm{d}I(z)}{\mathrm{d}z}$. [Check your results: (a) $I\left(\frac{3}{4}\right) = \ln 2$; (b) for $a = \frac{1}{2}$, $I\left(\frac{3}{2}\right) = \ln 2 + \frac{15}{16}$.]

(a) $I(z) = \int_0^z \mathrm{d}x \, \frac{1}{\sqrt{1+x^2}}$ (b) $I(z) = \int_0^z \mathrm{d}x \, \sqrt{1+a^2x^2}$.

Homework Problem 2: Vector space of complex numbers [3]

Show that the complex numbers $\mathbb{C}$ form a $\mathbb{R}$-vector space over the field of real numbers.

Homework Problem 3: Real vector space with unconventional composition rules [Bonus]

Points: (a)[1](M,Bonus); (b)[1](M,Bonus); (c)[1](E,Bonus)

For any $\mathbf{a} \in \mathbb{R}^2$, let $V_\mathbf{a} \equiv \{\mathbf{v_x}\}$ be a set whose elements $\mathbf{v_x}$, labelled by vectors $\mathbf{x} \in \mathbb{R}^2$, satisfy the following composition rules:

Addition: $\quad \boldsymbol{+} : \quad V_\mathbf{a} \times V_\mathbf{a} \to V_\mathbf{a}, \quad (\mathbf{v_x}, \mathbf{v_y}) \mapsto \mathbf{v_x} \boldsymbol{+} \mathbf{v_y} \equiv \mathbf{v_{x+y-a}}$

Multiplication by a scalar: $\quad \cdot : \quad \mathbb{R} \times V_{\mathbf{a}} \to V_{\mathbf{a}}, \qquad (\lambda, \mathbf{v_x}) \mapsto \lambda \cdot \mathbf{v_x} \equiv \mathbf{v}_{\lambda \mathbf{x} + f(\mathbf{a},\lambda)}$

Here $f(\mathbf{a}, \lambda)$ is a function of $\mathbf{a}$ and λ, whose form will be determined below.

(a) Show that $V_{\mathbf{a}}$, endowed with the composition rule $+$, forms an abelian group, and specify the neutral element of addition and the additive inverse of $\mathbf{v_x}$.

(b) Find the specific form of f that ensures that the triple $(V_{\mathbf{a}}, +, \cdot)$ forms an $\mathbb{R}$-vector space.

(c) Would a similar construction work for $\mathbf{a}$, $\mathbf{x} \in \mathbb{R}^n$ (with n a positive integer) instead of $\mathbb{R}^2$?

Homework Problem 4: Linear independence [3]
Points: (a)[2](M); (b)[1](M)

(a) Are the vectors $\mathbf{v}_1 = (1, 2, 3)^T$, $\mathbf{v}_2 = (2, 4, 6)^T$ and $\mathbf{v}_3 = (-1, -1, 0)^T$ linearly independent?

(b) Depending on whether your answer is yes or no, find a vector, $\mathbf{v}'_2$ such that $\mathbf{v}_1$, $\mathbf{v}'_2$ and $\mathbf{v}_3$ are linearly dependent or independent, respectively, and show explicitly that they have this property.

Homework Problem 5: Einstein summation convention [2]

Let $a_1 = 1$, $a_2 = 2$, $b^1 = -1$, $b^2 = x$. Evaluate the following expressions, formulated using the Einstein summation convention, as functions of x:

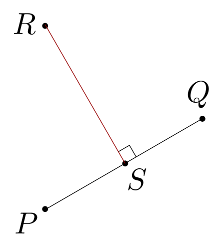
(a) $a_i b^i$, (b) $a_i a_j b^i b^j$, (c) $a_1 a_j b^2 b^j$.

[Check your results for $x = 3$: (a) 5, (b) 25, (c) 15.]

Homework Problem 6: Angle, orthogonal decomposition [3]
Points: (a)[0,5](E); (b)[1,5](E); (c)[1](E).

(a) Find the angle between the vectors $\mathbf{a} = (2, 0, \sqrt{2})^T$ and $\mathbf{b} = (\sqrt{2}, 1, 1)^T$.

In the figure, the points P, Q and R have coordinate vectors $\mathbf{p} = (-1, -1)^T$, $\mathbf{q} = (2, 1)^T$ and $\mathbf{r} = (-1, -1 + 13a)^T$, with a a positive real number. The line RS is perpendicular to the line PQ.



(b) Find the coordinate vector $\mathbf{s}$ of S, expressed as a function of a.
Hint: Let $\mathbf{c}$ denote the vector from P to Q, and $\mathbf{d}$ the vector from P to R, then decompose $\mathbf{d} = \mathbf{d}_{\|} + \mathbf{d}_{\perp}$ into components parallel and perpendicular to $\mathbf{c}$.

(c) Find the distance $\overline{RS}$ from R to S and the distance $\overline{PS}$ from P to S.

[Check your results for $a = 1$: (b) $\mathbf{s} = (5, 3)^T$, (c) $\overline{RS}^2 + \overline{PS}^2 = 169$.]

Homework Problem 7: Projection onto an orthonormal basis [2]
Points: (a)[1](E); (b)[1](E)

(a) Show that the vectors $\mathbf{e}'_1 = \frac{1}{9}(4, -1, 8)^T$, $\mathbf{e}'_2 = \frac{1}{9}(-7, 4, 4)^T$ and $\mathbf{e}'_3 = \frac{1}{9}(-4, -8, 1)^T$ form an orthonormal basis in $\mathbb{R}^3$.

(b) Let $\mathbf{w} = \mathbf{e}'_i w^i$ be the decomposition of $\mathbf{w} = (1, 2, 3)^T$ in this basis. Find the components w^i. [Check your results: $\sum_{i=1}^{3} w^i = \frac{22}{9}$.]

Homework Problem 8: Gram-Schmidt procedure [2]
Points: [2](E)

Apply the Gram-Schmidt procedure to the following set of linearly independent vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to construct an orthonormal set $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ with the same span and with $\mathbf{e}'_1 || \mathbf{v}_1$.

(a) $\mathbf{v}_1 = (0, 3, 0)^T$, $\mathbf{v}_2 = (1, -3, 0)^T$, $\mathbf{v}_3 = (2, 4, -2)^T$.

(b) $\mathbf{v}_1 = (-2, 0, 2)^T$, $\mathbf{v}_2 = (2, 1, 0)^T$, $\mathbf{v}_3 = (3, 6, 5)^T$.

Homework Problem 9: Non-orthogonal bases and metric [4]
Points: (a)[1](E); (b)[1](E); (c)[1](M); (d)[1](M)

Consider the vectors $\hat{\mathbf{v}}_1 = (2, 1, 2)^T$, $\hat{\mathbf{v}}_2 = (1, 0, 1)^T$, and $\hat{\mathbf{v}}_3 = (1, 1, 0)^T$, written as column vectors in the standard basis of $\mathbb{R}^3$. (In this problem we use the following notation: section **??**: vectors in the inner product space $\mathbb{R}^3$ carry a caret, e.g. $\hat{\mathbf{x}}$, and their components w.r.t. a given basis do not, e.g. $\mathbf{x}$.)

(a) Write the standard basis vector $\hat{\mathbf{e}}_1 = (1, 0, 0)^T$ as a linear combination of $\hat{\mathbf{v}}_1$, $\hat{\mathbf{v}}_2$ and $\hat{\mathbf{v}}_3$. Ditto for $\hat{\mathbf{e}}_2 = (0, 1, 0)^T$ and $\hat{\mathbf{e}}_3 = (0, 0, 1)^T$. Do $\hat{\mathbf{v}}_1$, $\hat{\mathbf{v}}_2$ and $\hat{\mathbf{v}}_3$ form a basis for $\mathbb{R}^3$?

(b) Let $\hat{\mathbf{x}} = \hat{\mathbf{v}}_1 x^1 + \hat{\mathbf{v}}_2 x^2 + \hat{\mathbf{v}}_3 x^3$ and $\hat{\mathbf{y}} = \hat{\mathbf{v}}_1 y^1 + \hat{\mathbf{v}}_2 y^2 + \hat{\mathbf{v}}_3 y^3$ be two vectors in $\mathbb{R}^3$, whose components w.r.t. $\hat{\mathbf{v}}_1$, $\hat{\mathbf{v}}_2$ and $\hat{\mathbf{v}}_3$ are given by $\mathbf{x} = (x^1, x^2, x^3) = (2, -5, 3)^T$ and $\mathbf{y} = (y^1, y^2, y^3) = (4, -1, -2)^T$, respectively. Express $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ as column vectors in the standard basis of $\mathbb{R}^3$ and compute their scalar product $\langle \hat{\mathbf{x}}, \hat{\mathbf{y}} \rangle_{\mathbb{R}^3}$.

(c) Find the components of the metric $g_{ij} = \langle \hat{\mathbf{v}}_i, \hat{\mathbf{v}}_j \rangle_{\mathbb{R}^3}$ explicitly.

(d) Now calculate the scalar product of $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ using the formula $\langle \hat{\mathbf{x}}, \hat{\mathbf{y}} \rangle_{\mathbb{R}^3} = \langle \mathbf{x}, \mathbf{y} \rangle_g = x^i g_{ij} y^j = x_j y^j$, with $x_j = x^i g_{ij}$, and carry out the sum over i and j explicitly. [Check: is the result consistent with that from (b)?]

[Total Points for Homework Problems: 22]
