Sheet 02: Vector Spaces, Euclidean Spaces

Solution Optional Problem 1: Vector space of real functions [2]

We have to verify that all the axioms for a vector space are satisfied. First, $(F, \boldsymbol{+})$ indeed has all the properties of an abelian group:

(i) Closure holds by definition: adding two functions from F again yields a function in F. ✓

(ii,v) Associativity and commutativity follow trivially from the corresponding properties of $\mathbb{R}$. For example associativity:

$$\Big[f \boldsymbol{+} [g \boldsymbol{+} h]\Big](x) = f(x) + [g \boldsymbol{+} h](x) = f(x) + \Big(g(x) + h(x)\Big)$$
$$= \Big(f(x) + g(x)\Big) + h(x) = [f \boldsymbol{+} g](x) + h(x) = \Big[[f \boldsymbol{+} g] \boldsymbol{+} h\Big](x)\,. \checkmark$$

(iii) The neutral element is the null function, defined by $f_{\text{null}} : x \mapsto f_{\text{null}}(x) \equiv 0$, since $f \boldsymbol{+} f_{\text{null}} : x \mapsto f(x) + f_{\text{null}}(x) = f(x) + 0 = f(x)$. ✓

(iv) The additive inverse of f is $-f$, defined by $-f : x \mapsto (-f)(x) \equiv -f(x)$, since $f \boldsymbol{+} (-f) : x \mapsto f(x) + (-f(x)) = 0$. ✓

Moreover, multiplication of any function with a scalar also has all the properties required for $(F, \boldsymbol{+}, \cdot)$ to be a vector space. Closure holds per definition. Furthermore:

(vi) Multiplication of a sum of scalars and a function is distributive:

$$\big[(\gamma + \lambda) \cdot f\big](x) = (\gamma + \lambda) f(x) = \gamma f(x) + \lambda f(x) = \big[\gamma \cdot f\big](x) + \big[\lambda \cdot f\big](x)$$
$$= \big[\gamma \cdot f \boldsymbol{+} \lambda \cdot f\big](x)\,. \checkmark$$

(vii) Multiplication of a scalar and a sum of functions is distributive:

$$\big[\lambda \cdot (f + g)\big](x) = \lambda\Big(\big[f \boldsymbol{+} g\big](x)\Big) = \lambda\Big(f(x) + g(x)\Big) = \lambda f(x) + \lambda g(x)$$
$$= \big[\lambda \cdot f\big](x) + \big[\lambda \cdot g\big](x) = \big[\lambda \cdot f \boldsymbol{+} \lambda \cdot g\big](x)\,. \checkmark$$

(viii) Multiplication of a product of scalars and a function is associative:

$$\big[(\gamma\lambda) \cdot f\big](x) = (\gamma\lambda) f(x) = \gamma\Big(\lambda f(x)\Big) = \gamma\big[\lambda \cdot f\big](x) = \big[\gamma \cdot (\lambda \cdot f)\big](x)\,. \checkmark$$

(ix) Neutral element: $\big[1 \cdot f\big](x) = 1 f(x) = f(x)$. ✓

Therefore, the triple $(F, \boldsymbol{+}, \boldsymbol{\cdot})$ is an $\mathbb{R}$-vector space.

Solution Optional Problem 2: Vector space of polynomials of degree n [3]

(a) The definition of addition of polynomials and the usual addition rule in $\mathbb{R}$ yield

$$\begin{aligned} p_{\mathbf{a}}(x) + p_{\mathbf{b}}(x) &= a_0x^0 + a_1x^1 + \dots a_nx^n + b_0x^0 + b_1x^1 + \dots b_nx^n \\ &= (a_0+b_0)x^0 + (a_1+b_1)x^1 + \dots (a_n+b_n)x^n = p_{\mathbf{a}+\mathbf{b}}(x) \ , \end{aligned}$$

since $\mathbf{a}+\mathbf{b} = (a_0+b_0, \dots, a_n+b_n)^T \in \mathbb{R}^{n+1}$. Therefore $\boxed{p_{\mathbf{a}} \boldsymbol{+} p_{\mathbf{b}} = p_{\mathbf{a}+\mathbf{b}}}$. ✓

The definition for the multiplication of a polynomial with a scalar and the usual multiplication rule in $\mathbb{R}$ yield

$$cp_{\mathbf{a}}(x) = c(a_0x^0 + a_1x^1 + \dots a_nx^n) = ca_0x^0 + ca_1x^1 + \dots ca_nx^n = p_{c\mathbf{a}}(x) \ ,$$

since $c\mathbf{a} = (ca_0, \dots, ca_n)^T \in \mathbb{R}^{n+1}$. Therefore $\boxed{c \boldsymbol{\cdot} p_{\mathbf{a}} = p_{c\mathbf{a}}}$. ✓

(b) We have to verify that all the axioms for a vector space are satisfied. First, $(P_n, \boldsymbol{+})$ indeed has all the properties of an abelian group:

(i) Closure: adding two polynomials of degree n again yields a polynomial of degree at most n. ✓

(ii,v) Associativity and commutativity follow trivially from the corresponding properties of $\mathbb{R}^{n+1}$. For example, consider associativity:

$$p_{\mathbf{a}} \boldsymbol{+} (p_{\mathbf{b}} \boldsymbol{+} p_{\mathbf{c}}) = p_{\mathbf{a}} \boldsymbol{+} p_{\mathbf{b}+\mathbf{c}} = p_{\mathbf{a}+(\mathbf{b}+\mathbf{c})} = p_{(\mathbf{a}+\mathbf{b})+\mathbf{c}} = p_{\mathbf{a}+\mathbf{b}} \boldsymbol{+} p_{\mathbf{c}} = (p_{\mathbf{a}} \boldsymbol{+} p_{\mathbf{b}}) \boldsymbol{+} p_{\mathbf{c}}. \checkmark$$

(iii) The neutral element is the null polynomial $p_{\mathbf{0}}$, i.e. the polynomial whose coefficients are all equal to 0. ✓

(iv) The additive inverse of $p_{\mathbf{a}}$ is $p_{-\mathbf{a}}$. ✓

Moreover, multiplication of any polynomial with a scalar also has all the properties required for $(P_n, \boldsymbol{+}, \boldsymbol{\cdot})$ to be a vector space. Multiplication with a scalar $c \in \mathbb{R}$ satisfies closure, since $c \boldsymbol{\cdot} p_{\mathbf{a}} = p_{c\mathbf{a}}$ again yields a polynomial of degree n. ✓ All the rules for multiplication by scalars follow directly from the corresponding properties of $\mathbb{R}^{n+1}$. ✓

Each element $p_{\mathbf{a}} \in P_n$ is uniquely identified by the element $\mathbf{a} \in \mathbb{R}^{n+1}$ – this identification is a bijection between P_n and $\mathbb{R}^{n+1}$, hence $(P_n, \boldsymbol{+}, \boldsymbol{\cdot})$ is isomorphic to $\mathbb{R}^{n+1}$ and has dimension $n+1$. ✓

(c) The bijection between P_n and $\mathbb{R}^{n+1}$ associates the standard basis vectors in $\mathbb{R}^{n+1}$, namely $\mathbf{e}_k = (0, \dots 1, \dots, 0)^T$ (with a 1 at position k and $0 \le k \le n$), with a basis in the vector space $(P_n, \boldsymbol{+}, \boldsymbol{\cdot})$, namely $\{p_{\mathbf{e}_0}, \dots, p_{\mathbf{e}_n}\}$, corresponding to the monomials $\{1, x, x^2, \dots, x^n\}$, since $p_{\mathbf{e}_k}(x) = x^k$. This statement corresponds to the obvious fact that every polynomial of degree n can be written as linear combination of monomials of degree $\le n$.

Solution Optional Problem 3: Unconventional inner products on $\mathbb{R}^2$ [2]

All the defining properties of an inner product are satisfied:

(i) Symmetric:

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_1 y_2 + x_2 y_1 + 3 x_2 y_2 = y_1 x_1 + y_1 x_2 + y_2 x_1 + 3 y_2 x_2 = \langle \mathbf{y}, \mathbf{x} \rangle \,. \checkmark$$

(ii,iii) Linear:

$$\begin{aligned}\langle \lambda \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle &= (\lambda x_1 + y_1) z_1 + (\lambda x_1 + y_1) z_2 + (\lambda x_2 + y_2) z_1 + 3(\lambda x_2 + y_2) z_2 \\ &= (\lambda x_1 z_1 + \lambda x_1 z_2 + \lambda x_2 z_1 + 3 \lambda x_2 z_2) + (y_1 z_1 + y_1 z_2 + y_2 z_1 + 3 y_2 z_2) \\ &= \lambda \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle \,. \checkmark\end{aligned}$$

(iii) Positive semi-definite:

$$\langle \mathbf{x}, \mathbf{x} \rangle = x_1 x_1 + x_1 x_2 + x_2 x_1 + 3 x_2 x_2 = (x_1 + x_2)^2 + 2 x_2^2 \ \geq \ 0 \,. \checkmark$$

If $\langle \mathbf{x}, \mathbf{x} \rangle$, then $\mathbf{x} = (0,0)^T$. ✓

Solution Optional Problem 4: Inner product and norm for the vector space of continuous functions [3]

(a) All the defining properties of an inner product are satisfied:

(i) Symmetric: $$\langle f, g \rangle = \int_I \mathrm{d}x \, f(x) g(x) = \int_I \mathrm{d}x \, g(x) f(x) = \langle g, f \rangle \,.\checkmark$$

(ii,iii) Linear: $$\begin{aligned}\langle \lambda \cdot f + g, h \rangle &= \int_I \mathrm{d}x \, (\lambda f(x) + g(x)) h(x) \\ &= \lambda \int_I \mathrm{d}x \, f(x) h(x) + \int_I \mathrm{d}x \, g(x) h(x) = \lambda \langle f, h \rangle + \langle g, h \rangle \,. \checkmark\end{aligned}$$

(iv) Positive semi-definite: $$\langle f, f \rangle = \int_I \mathrm{d}x f^2(x) \geq 0 \,.$$

Since the integrand is everywhere ≥ 0, the integral is also ≥ 0. ✓ Moreover, since f is continuous, the integral can equal 0 if and only if the integrand f^2 vanishes everywhere. Therefore $f(x) = 0$, i.e. f is the zero function. ✓

Optional: mathematical justification for the last statement: Suppose $f \neq 0$, then there exists an $x_0 \in I$ such that $(f(x_0))^2 \neq 0$. Since f is continuous, $(f(x))^2$ is non-zero in some neighbourhood of x_0, i.e., there exists a $\delta > 0$, such that for all $|x_0 - x| < \delta$, $|(f(x))^2| > \frac{1}{2} |(f(x_0))^2|$. Thus the integral must be larger than zero; e.g. we can find a lower bound as follows: $\langle f, f \rangle = \int_I \mathrm{d}x \, (f(x))^2 \geq \int_{x_0 - \delta}^{x_0 + \delta} \mathrm{d}x \, (f(x))^2 > \int_{x_0 - \delta}^{x_0 + \delta} \mathrm{d}x \, \frac{1}{2} (f(x_0))^2 = \delta f(x_0)^2 > 0$. Furthermore, for $f(x) \equiv 0$, we have $\langle f, f \rangle = \int_I \mathrm{d}x \, (f(x))^2 = \int_I \mathrm{d}x \, 0 = 0$. ✓

(b) $$\langle f_1, f_2 \rangle = \int_{-1}^{1} \mathrm{d}x \, f_1(x) f_2(x) = \int_{-1}^{1} \mathrm{d}x \, \sin\left(\frac{x}{\pi}\right) \cos\left(\frac{x}{\pi}\right) = \boxed{0} \,,$$

because the integrand is antisymmetric. Thus the two functions are 'orthogonal' to each other.

Explicitly, the substitution $u = \sin(x/\pi)$, $\mathrm{d}u = \mathrm{d}x \cos(x/\pi)/\pi$ gives:

$$\int_{-1}^{1} \mathrm{d}x\, \sin\left(\frac{x}{\pi}\right) \cos\left(\frac{x}{\pi}\right) = \pi \int_{-\sin\left(\frac{1}{\pi}\right)}^{\sin\left(\frac{1}{\pi}\right)} \mathrm{d}u\, u = \left.\frac{\pi u^2}{2}\right|_{-\sin\left(\frac{1}{\pi}\right)}^{\sin\left(\frac{1}{\pi}\right)} = 0\ .$$

[Total Points for Optional Problems: 10]
