



<https://moodle.lmu.de> → Kurse suchen: 'Rechenmethoden'

Sheet 02: Vector Spaces, Euclidean Spaces

(b)[2](E/M/A) means: problem (b) counts 2 points and is easy/medium hard/advanced

Suggestions for central tutorial: example problems 5, 7, 9, 8.

Videos exist for example problems 4 (L2.4.1), 9 (L3.3.7).

Optional Problem 1: Vector space of real functions [2]

Points: [2](M)

Let $F \equiv \{f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto f(x)\}$ be the set of real functions. Show that $(F, \boldsymbol{+}, \cdot)$ is an \mathbb{R} -vector space, where the addition of functions, and their multiplication by scalars, are defined as follows:

$$\boldsymbol{+} : F \times F \rightarrow F \quad (f, g) \mapsto f \boldsymbol{+} g, \quad \text{with} \quad f \boldsymbol{+} g : x \mapsto [f \boldsymbol{+} g](x) \equiv f(x) + g(x) \quad (1)$$

$$\cdot : \mathbb{R} \times F \rightarrow F \quad (\lambda, f) \mapsto \lambda \cdot f, \quad \text{with} \quad \lambda \cdot f : x \mapsto [\lambda \cdot f](x) \equiv \lambda f(x) \quad (2)$$

Remark regarding notation: It is important to distinguish the 'name' of a function, f , from the 'function value', $f(x)$, which it returns when evaluated at the argument x . The sum of the functions f and g is a function named $f \boldsymbol{+} g$. Equation (1) states that its function value at x , denoted by $[f \boldsymbol{+} g](x)$, is by definition equal to $f(x) + g(x)$, the sum of the function values of f and g at x . (For emphasis, in this problem we use square bracket to indicate the function name; elsewhere we'll use round brackets for this.) The product of the number c and the function f yields a function named $c \cdot f$. Eq. (2) states that its function value at x , denoted by $[c \cdot f](x)$, is by definition equal to $cf(x)$, the product of c with the function value of f at x .

Optional Problem 2: Vector space of polynomials of degree n [3]

Points: (a)[1](E); (b)[1](E); (c)[1](E)

The vector space of all real functions is infinite-dimensional. However, if only functions of a prescribed form are considered, the corresponding vector space can be finite-dimensional. As an example, it is shown in this problem that the set of all polynomials of degree n form a vector space of dimension $n + 1$, isomorphic to \mathbb{R}^{n+1} .

[Remark on the notation: In the context of the present problem on polynomials, x^k means "x to the power of k", and a_k is "the coefficient of x^k ". This is in contrast to the notation that we have adopted elsewhere when discussing vectors, where x^k stands for the k-th component of the vector $\mathbf{x} = \sum_k v_k x^k$ with respect to a basis of vectors $\{\mathbf{v}_k\}$. Every notational convention has exceptions!]

Let $p_{\mathbf{a}}$ denote a polynomial in the variable $x \in \mathbb{R}$ of degree n :

$$p_{\mathbf{a}} : \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto p_{\mathbf{a}}(x) \equiv a_0 x^0 + a_1 x^1 + \dots + a_n x^n.$$

$p_{\mathbf{a}}$ is uniquely specified by its $n + 1$ real coefficients a_0, a_1, \dots, a_n , which for notational brevity we arrange into a $(n + 1)$ -tuple, $\mathbf{a} = (a_0, a_1, \dots, a_n)^T \in \mathbb{R}^{n+1}$. Let $P_n = \{p_{\mathbf{a}} | \mathbf{a} \in \mathbb{R}^{n+1}\}$ denote the

set of all such polynomials of degree n . The natural definitions for adding such polynomials, or multiplying them by a scalar $c \in \mathbb{R}$, are:

$$\begin{aligned} p_{\mathbf{a}} + p_{\mathbf{b}} : \mathbb{R} &\rightarrow \mathbb{R}, & x &\mapsto p_{\mathbf{a}}(x) + p_{\mathbf{b}}(x), \\ c \cdot p_{\mathbf{a}} : \mathbb{R} &\rightarrow \mathbb{R}, & x &\mapsto c p_{\mathbf{a}}(x), \end{aligned}$$

where on the right side the usual addition and multiplication in \mathbb{R} is used.

(a) Show that the above addition and scalar multiplication rules imply the following composition rules in P_n :

$$\begin{aligned} \text{Addition of polynomials:} & \quad + : P_n \times P_n \rightarrow P_n, & (p_{\mathbf{a}}, p_{\mathbf{b}}) &\mapsto p_{\mathbf{a}} + p_{\mathbf{b}} \equiv p_{\mathbf{a}+\mathbf{b}}, \\ \text{Multiplication by a scalar:} & \quad \cdot : \mathbb{R} \times P_n \rightarrow P_n, & (c, p_{\mathbf{x}}) &\mapsto c \cdot p_{\mathbf{x}} \equiv p_{c\mathbf{a}}, \end{aligned}$$

where $\mathbf{a} + \mathbf{b}$ and $c\mathbf{a}$ denote the usual addition and scalar multiplication in \mathbb{R}^{n+1} .

(b) Show that $(P_n, +, \cdot)$ is an \mathbb{R} -vector space, and that it is isomorphic to \mathbb{R}^{n+1} .

(c) Find a set $n + 1$ of polynomials, $\{p_{\mathbf{a}_0}, \dots, p_{\mathbf{a}_n}\} \subset P_n$, forming a basis for this vector space.

Optional Problem 3: Unconventional inner products on \mathbb{R}^2 [2]

Points: [2](M)

The defining properties of an inner product on \mathbb{R}^n are of course satisfied not only by the 'standard' definition, $\langle \mathbf{x}, \mathbf{x} \rangle = \sum_{i=1}^n (x^i)^2$; there are infinitely many other bilinear forms that do so, too. The present problem illustrates this with a simple example. Show that the following map defines an inner product on the vector space \mathbb{R}^2 :

$$\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}, \quad (\mathbf{x}, \mathbf{y}) \mapsto x_1 y_1 + x_1 y_2 + x_2 y_1 + 3x_2 y_2.$$

Optional Problem 4: Inner product and norm for the vector space of continuous functions [3]

Points: (a)[2](M); (b)[1](M)

This problem illustrates a particularly important example of an inner product: in the space of continuous functions, an inner product can be defined via integration.

Let V be the vector space of *continuous* real functions defined on an interval $I \in \mathbb{R}$, $f : I \rightarrow \mathbb{R}$, with the usual composition rules of vector addition and scalar multiplication:

$$\begin{aligned} \forall f, g \in V : & \quad f + g : I \rightarrow \mathbb{R}, & x &\mapsto (f + g)(x) \equiv f(x) + g(x), \\ \forall f \in V, \lambda \in \mathbb{R} : & \quad \lambda \cdot f : I \rightarrow \mathbb{R}, & x &\mapsto (\lambda \cdot f)(x) \equiv \lambda (f(x)). \end{aligned}$$

(a) Show that the following map defines an inner product on V :

$$\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}, \quad (f, g) \mapsto \langle f, g \rangle \equiv \int_I dx f(x)g(x).$$

(b) Now consider $I = [-1, 1]$. Compute $\langle f_1, f_2 \rangle$ for $f_1(x) \equiv \sin\left(\frac{x}{\pi}\right)$ and $f_2(x) \equiv \cos\left(\frac{x}{\pi}\right)$.

[Total Points for Optional Problems: 10]
