Sheet 02: Vector Spaces, Euclidean Spaces

Solution Example Problem 1: $\sqrt{1-x^2}$ Integrals by trigonometric substitution [3]

(a) Since $\frac{\mathrm{d}}{\mathrm{d}x}\arcsin x = \frac{1}{\sqrt{1-x^2}}$, the primitive function of the integrand is known, and we may conclude immediately that $I(z) = [\arcsin x]_0^z = \arcsin z$.

Equivalently, we may compute the integral using the substitution $x = \sin y$, with $\mathrm{d}x = \mathrm{d}y\frac{\mathrm{d}x}{\mathrm{d}y} = \mathrm{d}y \sin' y = \mathrm{d}y\cos y$ and $\sqrt{1-x^2} = \sqrt{1-\sin^2 y} = \cos y$. The new integration boundaries are found by evaluating $y = \arcsin x$ at $x=0$ and $x=z$:

$$I(z) = \int_0^z \mathrm{d}x\,\frac{1}{\sqrt{1-x^2}} = \int_{\arcsin 0}^{\arcsin z}\mathrm{d}y\;\cos y\frac{1}{\cos y} = \int_0^{\arcsin z}\mathrm{d}y = \boxed{\arcsin z}.$$

Check result: $I\big(\frac{1}{\sqrt{2}}\big) = \arcsin\big(\frac{1}{\sqrt{2}}\big) = \frac{\pi}{4}$, since $\sin\big(\frac{\pi}{4}\big) = \frac{1}{\sqrt{2}}$. ✓

General check: $\dfrac{\mathrm{d}I(z)}{\mathrm{d}z} = \dfrac{\mathrm{d}}{\mathrm{d}z}\arcsin z = \boxed{\dfrac{1}{\sqrt{1-z^2}}}$. ✓

(b) We substitute $x = \frac{1}{a}\sin y$, with $\mathrm{d}x = \mathrm{d}y\frac{\mathrm{d}x}{\mathrm{d}y} = \mathrm{d}y\frac{1}{a}\cos y$ and $\sqrt{1-a^2x^2} = \cos y$:

$$I(z) = \int_0^z \mathrm{d}x\,\sqrt{1-a^2x^2} = \frac{1}{a}\int_{\arcsin 0}^{\arcsin(az)}\mathrm{d}y\;\cos y\cos y \equiv \frac{1}{a}\tilde{I}(b).$$

We compute the $\cos^2 y$ integral, with upper limit $b = \arcsin(az)$, by integrating by parts, with $u = \cos y$, $v = \sin y$, $u' = -\sin y$, $v' = \cos y$:

$$\tilde{I}(b) = \int_0^b \mathrm{d}y\;\overset{u}{\cos y}\,\overset{v'}{\cos y} \overset{uv-\int u'v}{=} \Big[\cos y\,\sin y\Big]_0^b - \int_0^b \mathrm{d}y\;\underbrace{[-\sin y]\sin y}_{\cos^2 y - 1}$$

$$= b + \cos b\sin b - \tilde{I}(b)$$

$$\Rightarrow \tilde{I}(b) = \tfrac{1}{2}\big[b + \sin b\cos b\big] = \tfrac{1}{2}\Big[b + \sin b\sqrt{1-\sin^2 b}\Big].$$

We expressed the r.h.s. through $\sin$, because the argument of $\tilde{I}(b)$ is $b = \arcsin(az)$.

$$\Rightarrow I(z) = \frac{1}{a}\tilde{I}\big(\arcsin(az)\big) = \boxed{\frac{1}{2a}\Big[\arcsin(az) + az\sqrt{1-a^2z^2}\Big]}.$$

Check result: for $a = \frac{1}{2}$, $I\big(\sqrt{2}\big) = \arcsin\big(\frac{1}{\sqrt{2}}\big) + \frac{1}{\sqrt{2}}\sqrt{1-\frac{1}{2}} = \frac{\pi}{4} + \frac{1}{2}$. ✓

General check: $\dfrac{\mathrm{d}I(z)}{\mathrm{d}z} \overset{\text{(a)}}{=} \dfrac{1}{2}\left[\dfrac{1}{\sqrt{1-a^2z^2}} + \sqrt{1-a^2z^2} + az\dfrac{-az}{\sqrt{1-a^2z^2}}\right] = \sqrt{1-a^2z^2}$. ✓

Solution Example Problem 2: Vector space axioms: rational numbers [3]

(a) First, we show that $(\mathbb{Q}^2, +)$ forms an abelian group.

(i) Closure holds by definition. ✓

(ii) Associativity:
$$\left[\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} + \begin{pmatrix} y^1 \\ y^2 \end{pmatrix}\right] + \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} = \begin{pmatrix} x^1 + y^1 \\ x^2 + y^2 \end{pmatrix} + \begin{pmatrix} z^1 \\ z^2 \end{pmatrix}$$
$$= \begin{pmatrix} x^1 + y^1 + z^1 \\ x^2 + y^2 + z^2 \end{pmatrix} = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} + \begin{pmatrix} y^1 + z^1 \\ y^2 + z^2 \end{pmatrix}$$
$$= \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} + \left[\begin{pmatrix} y^1 \\ y^2 \end{pmatrix} + \begin{pmatrix} z^1 \\ z^2 \end{pmatrix}\right]. \checkmark$$

(iii) Neutral element: $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the neutral element. ✓

(iv) Additive inverse: $\begin{pmatrix} -x^1 \\ -x^2 \end{pmatrix} \in \mathbb{Q}^2$ is the additive inverse of $\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \in \mathbb{Q}^2$. ✓

(v) Commutativity: follows (component-wise) from the commutativity of $\mathbb{Q}$. ✓

Second, we show that scalar multiplication, $\cdot$, likewise has the properties required for $(\mathbb{Q}^2, +, \cdot)$ to form a vector space. Since the product of two rational numbers is always rational, $\left(\frac{p_1}{q_1} \cdot \frac{p_2}{q_2} = \frac{(p_1 p_2)}{(q_1 q_2)}\right)$, closure holds by definition. Moreover:

(vi) Multiplication of a sum of scalars and a vector is distributive:
$$(\lambda + \mu) \cdot \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} (\lambda + \mu) x^1 \\ (\lambda + \mu) x^2 \end{pmatrix} = \begin{pmatrix} \lambda x^1 + \mu x^1 \\ \lambda x^2 + \mu x^2 \end{pmatrix} = \lambda \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} + \mu \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}. \checkmark$$

(vii) Multiplication of a scalar and a sum of vectors is distributive:
$$\lambda \cdot \left[\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} + \begin{pmatrix} y^1 \\ y^2 \end{pmatrix}\right] = \begin{pmatrix} \lambda x^1 + \lambda y^1 \\ \lambda x^2 + \lambda y^2 \end{pmatrix} = \lambda \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} + \lambda \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} . \checkmark$$

(viii) Multiplication of a product of scalars and a vector is associative:
$$(\lambda \mu) \cdot \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} \lambda \mu x^1 \\ \lambda \mu x^2 \end{pmatrix} = \lambda \left[\mu \cdot \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}\right] . \checkmark$$

(ix) Neutral element: $1 \cdot \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$. ✓

Therefore, the triple $(\mathbb{Q}^2, +, \cdot)$ represents a $\mathbb{Q}$-vector space.

(b) The set of integers $\mathbb{Z}$ does not form a field, since not for each $a \in \mathbb{Z} \setminus \{0\}$ a multiplicative inverse $a^{-1} \in \mathbb{Z}$ exists (e.g. the equation $2 \cdot a = 1$ has no solution within the integers). Hence, it is also *not* possible to construct any vector space over the integers.

Solution Example Problem 3: Real vector space with unconventional composition rules [Bonus]

First, we show that $(V_a, \boldsymbol{+})$ forms an abelian group.

(i) Closure holds by definition. ✓

(ii) Associativity: $$(\mathbf{v}_x \boldsymbol{+} \mathbf{v}_y) \boldsymbol{+} \mathbf{v}_z = \mathbf{v}_{x+y+a} \boldsymbol{+} \mathbf{v}_z = \mathbf{v}_{(x+y+a)+z+a} = \mathbf{v}_{x+y+z+2a}$$ $$= \mathbf{v}_{x+(y+z+a)+a} = \mathbf{v}_x \boldsymbol{+} \mathbf{v}_{y+z+a} = \mathbf{v}_x \boldsymbol{+} (\mathbf{v}_y \boldsymbol{+} \mathbf{v}_z) \, . \checkmark$$

(iii) Neutral element: $\mathbf{v}_x \boldsymbol{+} \mathbf{v}_{-a} = \mathbf{v}_{x+(-a)+a} = \mathbf{v}_x \, , \quad \Rightarrow \quad \mathbf{0} = \mathbf{v}_{-a} \, . \checkmark$

(iv) Additive inverse: $\mathbf{v}_x \boldsymbol{+} \mathbf{v}_{-x-2a} = \mathbf{v}_{x+(-x-2a)+a} = \mathbf{v}_{-a} = \mathbf{0} \, , \Rightarrow -\mathbf{v}_x = \mathbf{v}_{-x-2a} \, . \checkmark$

(v) Commutativity : $\mathbf{v}_x \boldsymbol{+} \mathbf{v}_y = \mathbf{v}_{x+y+a} = \mathbf{v}_{y+x+a} = \mathbf{v}_y \boldsymbol{+} \mathbf{v}_x \, . \checkmark$

Second, we show that scalar multiplication, $\boldsymbol{\cdot}$, likewise has the properties required for $(V_a, \boldsymbol{+}, \boldsymbol{\cdot})$ to form a vector space. Closure holds by definition. Moreover:

(vi) Multiplication of a sum of scalars and a vector is distributive:

$$\begin{aligned}(\gamma + \lambda) \boldsymbol{\cdot} \mathbf{v}_x &= \mathbf{v}_{(\gamma+\lambda)x + a(\gamma+\lambda-1)} = \mathbf{v}_{\gamma x + a(\gamma-1) + \lambda x + a(\lambda-1) + a} \\ &= \mathbf{v}_{\gamma x + a(\gamma-1)} \boldsymbol{+} \mathbf{v}_{\lambda x + a(\lambda-1)} = \gamma \boldsymbol{\cdot} \mathbf{v}_x \boldsymbol{+} \lambda \boldsymbol{\cdot} \mathbf{v}_x \, . \checkmark\end{aligned}$$

(vii) Multiplication of a scalar and a sum of vectors is distributive:

$$\begin{aligned}\lambda \boldsymbol{\cdot} (\mathbf{v}_x + \mathbf{v}_y) &= \lambda \boldsymbol{\cdot} \mathbf{v}_{x+y+a} = \mathbf{v}_{\lambda(x+y+a) + a(\lambda-1)} = \mathbf{v}_{\lambda x + a(\lambda-1) + \lambda y + a(\lambda-1) + a} \\ &= \mathbf{v}_{\lambda x + a(\lambda-1)} \boldsymbol{+} \mathbf{v}_{\lambda y + a(\lambda-1)} = \lambda \boldsymbol{\cdot} \mathbf{v}_x \boldsymbol{+} \lambda \boldsymbol{\cdot} \mathbf{v}_y \, . \checkmark\end{aligned}$$

(viii) Multiplication of a product of scalars and a vector is associative:

$$(\gamma\lambda) \boldsymbol{\cdot} \mathbf{v}_x = \mathbf{v}_{(\gamma\lambda)x + a(\gamma\lambda-1)} = \mathbf{v}_{\gamma(\lambda x + a(\lambda-1)) + a(\gamma-1)} = \gamma \boldsymbol{\cdot} \mathbf{v}_{\lambda x + a(\lambda-1)} = \gamma \boldsymbol{\cdot} (\lambda \boldsymbol{\cdot} \mathbf{v}_x) \, . \checkmark$$

(ix) Neutral element: $1 \boldsymbol{\cdot} \mathbf{v}_x = \mathbf{v}_{x+a(1-1)} = \mathbf{v}_x \, . \checkmark$

Therefore, the triple $(V_a, \boldsymbol{+}, \boldsymbol{\cdot})$ represents an $\mathbb{R}$-vector space.

Solution Example Problem 4: Linear independence [3]

(a) The three vectors are linearly independent if and only if the only solution to the equation

$$\mathbf{0} = a^1 \mathbf{v}_1 + a^2 \mathbf{v}_2 + a^3 \mathbf{v}_3 = a^1 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + a^2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + a^3 \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} , \quad \text{with} \quad a^j \in \mathbb{R}, \tag{1}$$

is the trivial one, $a^1 = a^2 = a^3 = 0$. The vector equation (1) yields a system of three equations, (i)-(iii), one for each of the three components of (1), which we solve as follows:

$$\begin{aligned}
&\text{(i)} \quad 0a^1 + 1a^2 + 2a^3 = 0 \quad &&\overset{\text{(i)}}{\Rightarrow} \quad &&\text{(iv)} \quad \boxed{a^2 = -2a^3} \\
&\text{(ii)} \quad 1a^1 - 1a^2 - 1a^3 = 0 \quad &&\overset{\text{(iv) in (ii)}}{\Rightarrow} \quad &&\text{(v)} \quad \boxed{a^1 = -a^3} \\
&\text{(iii)} \quad 2a^1 + 1a^2 + 4a^3 = 0 \quad &&\overset{\text{(iv,v) in (iii)}}{\Rightarrow} \quad &&\text{(vi)} \quad 0 = 0
\end{aligned}$$

(i) yields (iv): $a^2 = -2a^3$. (iv) inserted into (ii) yields (v): $a^1 = -a^3$. Inserting (iv) and (v) into (iii) yields no new information. There are thus infinitely many non-trivial solutions (one for every value of $a^3 \in \mathbb{R}$), hence $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are $\boxed{\text{not}}$ linearly independent.

(b) The desired vector $\mathbf{v}_2' = (x, y, z)^T$ should be linearly independent from $\mathbf{v}_1$ and $\mathbf{v}_3$, i.e. its components x, y and z should be chosen such that the equation $\mathbf{0} = a^1\mathbf{v}_1 + a^2\mathbf{v}_2' + a^3\mathbf{v}_3$ has no non-trivial solution, i.e. that it implies $a^1 = a^2 = a^3 = 0$:

$$\begin{array}{llcll}
\text{(i)} & 0a^1 + xa^2 + 2a^3 = 0 & \overset{\text{(i)}}{\Rightarrow} & \text{(iv)} & \text{choose } \boxed{x = 0}\text{, then } a^3 = 0. \\
\text{(ii)} & 1a^1 + ya^2 - 1a^3 = 0 & \overset{\text{(iv) in (ii)}}{\Rightarrow} & \text{(v)} & \text{choose } \boxed{y = 0}\text{, then } a^1 = 0. \\
\text{(iii)} & 2a^1 + za^2 + 4a^3 = 0 & \overset{\text{(iv),(v) in (iii)}}{\Rightarrow} & \text{(vi)} & \text{choose } \boxed{z = 1}\text{, then } a^2 = 0.
\end{array}$$

(i) yields (iv): $2a^3 = -xa^2$; to enforce $a^3 = 0$ we choose $x = 0$. (iv) inserted into (ii) yields (v): $a^1 = -ya^2$; to enforce $a^1 = 0$ we choose $y = 0$. (iv,v) inserted into (iii) yields $za^2 = 0$; to enforce $a^2 = 0$ we choose $z = 1$. Thus $\boxed{\mathbf{v}_2' = (0, 0, 1)^T}$ is a choice for which $\mathbf{v}_1$, $\mathbf{v}_2'$ are $\mathbf{v}_3$ linearly indepedent. This choice is not unique – there are infinitely many alternatives; one of them, e.g. is $\mathbf{v}_2' = (0, 1, 0)^T$.

Solution Example Problem 5: Einstein summation convention [2]

(a) $a_i b^i = b^j a_j$ is $\boxed{\text{true}}$, since i and j are dummy variables which are summed over, hence we may rename as we please:

$$a_i b^i = \sum_{i=1}^{2} a_i b^i = a_1 b^1 + a_2 b^2 = b^1 a_1 + b^2 a_2 = \sum_{j=1}^{2} b^j a_j = b^j a_j \,. \checkmark$$

(b) $a_i \delta^i{}_j b^j = a_k b^k$ is $\boxed{\text{true}}$, since $\delta^i{}_j$ is nonzero only for $i = j$, in which case it equals 1:

$$a_i \delta^i{}_j b^j = a_1 \underbrace{(\delta^1{}_1)}_{=1} b^1 + a_1 \underbrace{(\delta^1{}_2)}_{=0} b^2 + a_2 \underbrace{(\delta^2{}_1)}_{=0} b^1 + a_2 \underbrace{(\delta^2{}_2)}_{=1} b^2 = a_1 b^1 + a_2 b^2 = a_k b^k \,. \checkmark$$

(c) $a_i b^j a_j b^k \overset{?}{=} a_k b^l a_l b^i$ is $\boxed{\text{false}}$, since the indices i and k are *not* repeated, i.e. they are not summed over and hence may not renamed. For example, for $i = 1$ and $k = 2$ the left-hand side, $a_1(b^1 a_1 + b^2 a_2) b^2$, clearly differs from right-hand side, $a_2(b^1 a_1 + b^2 a_2) b^1$.

(d) $a_1 a_i b^1 b^i + b^2 a_j a_2 b^j = (a_i b^i)^2$ is $\boxed{\text{true}}$, since multiplication is associative and commutative and we may rename dummy indices as we please:

$$a_1 a_i b^1 b^i + b^2 a_j a_2 b^j = a_1 b^1 a_i b^i + a_2 b^2 a_i b^i = (a_1 b^1 + a_2 b^2)(a_i b^i) = (a_j b^j)(a_i b^i) = (a_i b^i)^2. \checkmark$$

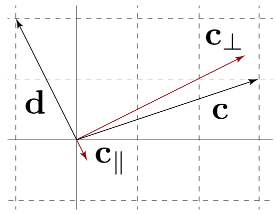
In practice, the arguments illustrated above need not be written out explicitly. Relations such as (a), (b) and (d) may be simply written down without further discussion.

Solution Example Problem 6: Angle, orthogonal decomposition [2]

(a) $$\cos(\angle(\mathbf{a},\mathbf{b})) = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \frac{3\cdot 7 + 4\cdot 1}{\sqrt{9+16}\cdot\sqrt{49+1}} = \frac{1}{\sqrt{2}} \quad\Rightarrow\quad \angle(\mathbf{a},\mathbf{b}) = \boxed{\frac{\pi}{4}}$$

(b) $$\mathbf{c}_{\|} = \frac{(\mathbf{c}\cdot\mathbf{d})\mathbf{d}}{\|\mathbf{d}\|^2} = \frac{3\cdot(-1)+1\cdot 2}{1+4}\begin{pmatrix}-1\\2\end{pmatrix} = \boxed{\frac{1}{5}\begin{pmatrix}1\\-2\end{pmatrix}}$$

$$\mathbf{c}_{\perp} = \mathbf{c} - \mathbf{c}_{\|} = \begin{pmatrix}3\\1\end{pmatrix} - \frac{1}{5}\begin{pmatrix}1\\-2\end{pmatrix} = \boxed{\frac{7}{5}\begin{pmatrix}2\\1\end{pmatrix}}$$



Consistency check: $\mathbf{c}_\perp \cdot \mathbf{c}_\| = \frac{1}{25}\big(1\cdot 14 - 2\cdot 7\big) = 0.$ ✓

Solution Example Problem 7: Projection onto an orthonormal basis [2]

(a) $$\langle \mathbf{e}'_1, \mathbf{e}'_1\rangle = \tfrac{1}{2}\big[1\cdot 1 + 1\cdot 1\big] = 1\,, \qquad \langle \mathbf{e}'_1, \mathbf{e}'_2\rangle = \tfrac{1}{2}\big[1\cdot 1 + (-1)\cdot 1\big] = 0\,.$$
$$\langle \mathbf{e}'_2, \mathbf{e}'_2\rangle = \tfrac{1}{2}\big[1\cdot 1 + (-1)\cdot(-1)\big] = 1$$

The two vectors are normalized and orthogonal to each other, $\boxed{\langle \mathbf{e}'_i, \mathbf{e}'_j\rangle = \delta_{ij}}$, therefore they form an orthonormal basis of $\mathbb{R}^2$. ✓

(b) Since the vectors $\{\mathbf{e}'_1, \mathbf{e}'_2\}$ form an orthonormal basis, the component w^i of the vector $\mathbf{w} = (-2,3)^T = \mathbf{e}'_i w^i$ with respect to this basis is given by the projection $w^i = \langle \mathbf{e}'^i, \mathbf{w}\rangle$ (with $\mathbf{e}'^i = \mathbf{e}'_i$):

$$w^1 = \langle \mathbf{e}'^1, \mathbf{w}\rangle = \tfrac{1}{\sqrt{2}}\big[1\cdot(-2) + 1\cdot 3\big] = \boxed{\tfrac{1}{\sqrt{2}}},$$
$$w^2 = \langle \mathbf{e}'^2, \mathbf{w}\rangle = \tfrac{1}{\sqrt{2}}\big[1\cdot(-2) - 1\cdot 3\big] = \boxed{-\tfrac{5}{\sqrt{2}}}.$$

Solution Example Problem 8: Gram-Schmidt procedure [2]

Strategy: iterative orthogonalization and normalization, starting from $\mathbf{v}_{1,\perp} = \mathbf{v}_1$:

Starting vector: $\mathbf{v}_{1,\perp} = \mathbf{v}_1 = (1,-2,1)^T$

Normalizing $\mathbf{v}_{1,\perp}$: $$\mathbf{e}'_1 = \frac{\mathbf{v}_{1,\perp}}{\|\mathbf{v}_{1,\perp}\|} = \boxed{\tfrac{1}{\sqrt{6}}(1,-2,1)^T} = \mathbf{e}'^1\,.$$

Orthogonalizing $\mathbf{v}_2$: $\mathbf{v}_{2,\perp} = \mathbf{v}_2 - \mathbf{e}'_1\langle \mathbf{e}'^1, \mathbf{v}_2\rangle = (1,1,1)^T - \mathbf{e}'_1(0)$

Normalizing $\mathbf{v}_{2,\perp}$: $$\mathbf{e}'_2 = \frac{\mathbf{v}_{2,\perp}}{\|\mathbf{v}_{2,\perp}\|} = \boxed{\tfrac{1}{\sqrt{3}}(1,1,1)^T} = \mathbf{e}'^2\,.$$

Orthogonalizing $\mathbf{v}_3$: $$\mathbf{v}_{3,\perp} = \mathbf{v}_3 - \mathbf{e}'_1\langle \mathbf{e}'^1, \mathbf{v}_3\rangle - \mathbf{e}'_2\langle \mathbf{e}'^2, \mathbf{v}_3\rangle$$
$$= (0,1,2)^T - \mathbf{e}'_1(0) - \tfrac{1}{\sqrt{3}}(1,1,1)^T\left(3\tfrac{1}{\sqrt{3}}\right) = (-1,0,1)^T$$

Normalizing $\mathbf{v}_{3,\perp}$: $$\mathbf{e}'_3 = \frac{\mathbf{v}_{3,\perp}}{\|\mathbf{v}_{3,\perp}\|} = \boxed{\tfrac{1}{\sqrt{2}}(-1,0,1)^T} = \mathbf{e}'^3\,.$$

Solution Example Problem 9: Non-orthogonal basis and metric [4]

(a) $\hat{\mathbf{v}}_1 = \begin{pmatrix}2\\0\end{pmatrix}, \quad \hat{\mathbf{v}}_2 = \begin{pmatrix}1\\1\end{pmatrix}; \qquad \Rightarrow \qquad \hat{\mathbf{e}}_1 = \begin{pmatrix}1\\0\end{pmatrix} = \boxed{\tfrac{1}{2}\hat{\mathbf{v}}_1},\ \hat{\mathbf{e}}_2 = \begin{pmatrix}0\\1\end{pmatrix} = \boxed{-\tfrac{1}{2}\hat{\mathbf{v}}_1 + \hat{\mathbf{v}}_2}.$

The vectors $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$ form a basis, because both standard basis vectors $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ can be written in terms of them.

(b) A representation of the vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ as column vectors in the standard basis of $\mathbb{R}^2$ can be found as follows:

$$\hat{\mathbf{x}} = \hat{\mathbf{v}}_1 x^1 + \hat{\mathbf{v}}_2 x^2, \quad x^1 = 3,\ x^2 = -4 \qquad \Rightarrow \qquad \hat{\mathbf{x}} = \begin{pmatrix}2\\0\end{pmatrix}3 + \begin{pmatrix}1\\1\end{pmatrix}(-4) = \boxed{\begin{pmatrix}2\\-4\end{pmatrix}}.$$

$$\hat{\mathbf{y}} = \hat{\mathbf{v}}_1 y^1 + \hat{\mathbf{v}}_2 y^2, \quad y^1 = -1,\ y^2 = 3 \qquad \Rightarrow \qquad \hat{\mathbf{y}} = \begin{pmatrix}2\\0\end{pmatrix}(-1) + \begin{pmatrix}1\\1\end{pmatrix}3 = \boxed{\begin{pmatrix}1\\3\end{pmatrix}}.$$

Scalar product: $\langle \hat{\mathbf{x}}, \hat{\mathbf{y}} \rangle_{\mathbb{R}^2} = \begin{pmatrix}2\\-4\end{pmatrix} \cdot \begin{pmatrix}1\\3\end{pmatrix} = 2\cdot 1 + (-4)\cdot 3 = \boxed{-10}.$

(c)
$$g_{11} = \langle \hat{\mathbf{v}}_1, \hat{\mathbf{v}}_1 \rangle_{\mathbb{R}^2} = \boxed{4}, \qquad g_{12} = \langle \hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2 \rangle_{\mathbb{R}^2} = \boxed{2},$$
$$g_{21} = \langle \hat{\mathbf{v}}_2, \hat{\mathbf{v}}_1 \rangle_{\mathbb{R}^2} = \boxed{2}, \qquad g_{22} = \langle \hat{\mathbf{v}}_2, \hat{\mathbf{v}}_2 \rangle_{\mathbb{R}^2} = \boxed{2}.$$

(d)
$$\begin{aligned}\langle \hat{\mathbf{x}}, \hat{\mathbf{y}} \rangle_{\mathbb{R}^2} &= \langle \mathbf{x}, \mathbf{y} \rangle_g = x^i g_{ij} y^j \\ &= 3\cdot 4\cdot(-1) + 3\cdot 2\cdot 3 + (-4)\cdot 2\cdot(-1) + (-4)\cdot 2\cdot 3 = \boxed{-10}.\ \checkmark\ \text{[= (b)]}\end{aligned}$$

[Total Points for Example Problems: 21]
