



<https://moodle.lmu.de> → Kurse suchen: 'Rechenmethoden'

## Sheet 01: Mathematical Foundations

(b)[2](E/M/A) means: problem (b) counts 2 points and is easy/medium hard/advanced  
Suggestions for central tutorial: example problems 9, 10, 4, 3.  
Videos exist for example problems 9 (C2.3.1), 10 (C2.3.3).

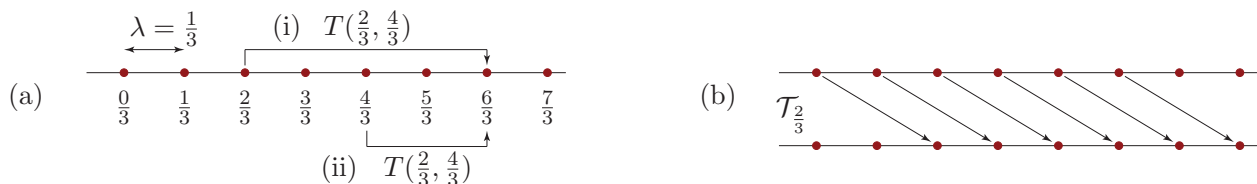
### Optional Problem 1: Group of discrete translations in one dimension [4]

Points: (a)[2](E); (b)[2](M)

In this problem we show that discrete translations on an infinite, one-dimensional lattice form a group. Let us denote the lattice constant, i.e. the fixed distance between neighboring lattice points, by  $\lambda \in \mathbb{R}^+$ , a positive, real number. The lattice  $\mathbb{G}$  consists of the set of all integer multiples of  $\lambda$ , i.e.  $\mathbb{G} \equiv \lambda\mathbb{Z} \equiv \{x \in \mathbb{R} | \exists n \in \mathbb{Z} : x = \lambda \cdot n\}$ , where  $\cdot$  is the usual multiplication rule in  $\mathbb{R}$ . Note that for any given  $x \in \mathbb{G}$ ,  $n$  is uniquely determined. On this lattice we define 'translation' by the group operation

$$T : \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{G}, \quad (x, y) \mapsto T(x, y) \equiv x + y,$$

where  $+$  denotes the usual addition of real numbers. Since this operation is symmetric, it can be visualized in two equivalent ways:  $T(x, y)$  describes (i) a 'shift' or a 'translation' of lattice point  $x$  by the distance  $y$ , or (ii) a translation of lattice point  $y$  by the distance  $x$ . [Figure (a), where  $\lambda = \frac{1}{3}$ , shows both visualizations of  $T(\frac{2}{3}, \frac{4}{3})$ .]



(a) Show that  $(\mathbb{G}, T)$  forms an abelian group.

(b) For a given  $y \in \mathbb{G}$  we now define, in accordance with visualization (i), a 'translation' of the lattice by  $y$ , i.e. each lattice point  $x$  is 'shifted' by  $y$ :

$$\mathcal{T}_y : \mathbb{G} \rightarrow \mathbb{G}, \quad x \mapsto \mathcal{T}_y(x) \equiv T(x, y).$$

[Figure (b), where  $\lambda = \frac{1}{3}$ , shows  $\mathcal{T}_{\frac{2}{3}}$ .] Now consider the set of all such translations,  $\mathbb{T} \equiv \{\mathcal{T}_y, y \in \mathbb{G}\}$ . Show that  $(\mathbb{T}, +)$  forms an abelian group, where  $+$  is defined as

$$+ : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{T}, \quad (\mathcal{T}_x, \mathcal{T}_y) \mapsto \mathcal{T}_x + \mathcal{T}_y \equiv \mathcal{T}_{T(x, y)}.$$

Remark: the set  $\mathbb{T}$  underlying this group consists of maps (namely translations), illustrating that the set underlying a group need not be 'simple'.

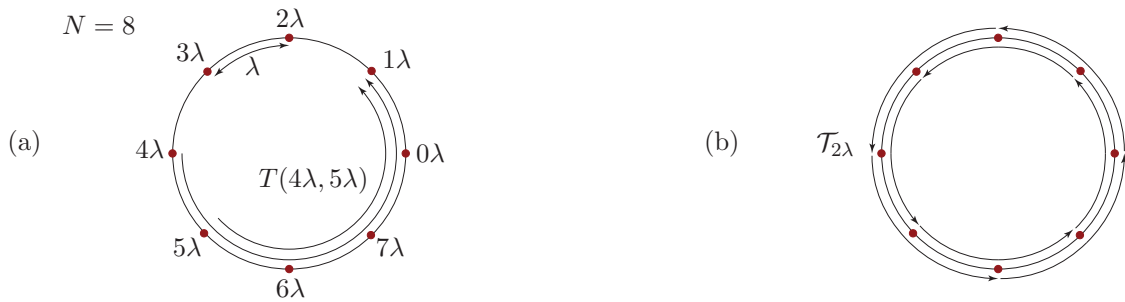
### Optional Problem 2: Group of discrete translations on a ring [4]

Points: (a)[2](M); (b)[2](M)

In this problem we show that discrete translations on a finite, one-dimensional lattice with periodic boundary conditions form a group. Consider a ring with radius  $0 < R \in \mathbb{R}$  and lattice constant  $\lambda = 2\pi R/N$  with  $N \in \mathbb{N}$ , thus  $\mathbb{G} \equiv \lambda(\mathbb{Z} \bmod N) \equiv \{x \in \mathbb{R} | \exists n \in \{0, 1, \dots, N-1\} : x = \lambda \cdot n\}$ , where  $\cdot$  is the usual multiplication rule in  $\mathbb{R}$ . Note that for any given  $x \in \mathbb{G}$ ,  $n$  is uniquely determined. The ring forms a ‘periodic’ structure: when counting its sites,  $0\lambda$  and  $N\lambda$  describe the same lattice site, the same is true for  $1\lambda$  and  $(1+N)\lambda$ , for  $2\lambda$  and  $(2+N)\lambda$ , etc. On this lattice we define a group operation, corresponding to a ‘translation’, using addition modulo  $N$ :

$$T : \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{G}, \quad (x, y) = (\lambda \cdot n_x, \lambda \cdot n_y) \mapsto T(x, y) \equiv \lambda \cdot ((n_x + n_y) \bmod N).$$

Here  $+$  is the usual addition of integers, and  $n \bmod N$  (spoken as ‘ $n \bmod N$ ’) is defined as the integer remainder after division of  $n$  by  $N$  (e.g.  $9 \bmod 8 = 1$ ). [For  $N = 8$ , figure (a) shows two visualizations of the translation  $T(4\lambda, 5\lambda)$ : as a ‘shift’ of the lattice site  $4\lambda$  by the distance  $5\lambda$  along the ring, or of the site  $5\lambda$  by the distance  $4\lambda$ .]



(a) Show that  $(\mathbb{G}, T)$  forms an abelian group.

(b) For a given  $y \in \mathbb{G}$  we now define a ‘translation’ of the lattice by  $y$ ,

$$\mathcal{T}_y : \mathbb{G} \rightarrow \mathbb{G}, \quad x \mapsto \mathcal{T}_y(x) \equiv T(x, y)$$

i.e. each site  $x$  is ‘shifted’ by  $y$  along the ring. [For  $N = 8$ , figure (b) shows the translation  $\mathcal{T}_{2\lambda}$ ]. Now consider the set of all such translations,  $\mathbb{T} \equiv \{\mathcal{T}_y, y \in \mathbb{G}\}$ . Show that  $(\mathbb{T}, +)$  forms an abelian group, where the group operation  $+$  is defined as

$$+ : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{T}, \quad (\mathcal{T}_x, \mathcal{T}_y) \mapsto \mathcal{T}_x + \mathcal{T}_y \equiv \mathcal{T}_{T(x,y)}.$$

### Optional Problem 3: L’Hôpital’s rule [4]

Points: (a)[0,5](E); (b)[0,5](E); (c)[1](M); (d)[1](M); (e)[1](M)

Consider the following question: what is the limiting value of the ratio,  $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)}$ , if the functions  $f$  and  $g$  both vanish at the point  $x_0$ ? The naive answer,  $\frac{f(x_0)}{g(x_0)} \stackrel{?}{=} \frac{0}{0}$ , is ill-defined. However, if both functions have a finite slope at  $x_0$ , we may use a linear approximation for both,  $f(x_0 + \delta) \simeq 0 + \delta f'(x_0)$  and  $g(x_0 + \delta) \simeq 0 + \delta g'(x_0)$ , to obtain  $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}$ . This result is a special case of L’Hôpital’s rule.

The general formulation of L’Hôpital’s rule is: If either  $\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} g(x) = 0$  or  $\lim_{x \rightarrow x_0} |f(x)| = \lim_{x \rightarrow x_0} |g(x)| = \infty$ , and the limit  $\lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}$  exists, then

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}. \quad (1)$$

The proof of this general statement is non-trivial, but is a standard topic in calculus textbooks.

Use L'Hôpital's rule to evaluate the following limits as functions of the real number  $a$ : [Check your results against those in square brackets, where  $[a, b]$  means that the limit  $L(a) = b$ .]

$$(a) \lim_{x \rightarrow 1} \frac{x^2 + (a-1)x - a}{x^2 + 2x - 3} \quad [3, 1] \qquad (b) \lim_{x \rightarrow 0} \frac{\sin(ax)}{x + ax^2} \quad [2, 2]$$

If not only  $f$  and  $g$  but also  $f'$  and  $g'$  all vanish at  $x_0$ , the limit on the r.h.s. of L'Hôpital's rule may be evaluated by applying the rule a second time (or  $n + 1$  times, if the derivatives up to  $f^{(n)}$  and  $g^{(n)}$  all vanish at  $x_0$ ). Use this procedure to evaluate the following limits:

$$(c) \lim_{x \rightarrow 0} \frac{1 - \cos(ax)}{\sin^2 x}, \quad [4, 8] \qquad (d) \lim_{x \rightarrow 0} \frac{x^3}{\sin(ax) - ax}. \quad [2, -\frac{3}{4}]$$

(e) Use L'Hôpital's rule to show that  $\lim_{x \rightarrow 0} (x \ln x) = 0$  (with  $x > 0$ ). This result implies that for  $x \rightarrow 0$ , ' $x$  decreases more quickly than  $\ln(x)$  diverges', i.e. 'linear beats log'.

**Optional Problem 4: L'Hôpital's rule [4]**

Points: (a)[0,5](E); (b)[0,5](E); (c)[1](M); (d)[1](M); (e)[1](M)

Use L'Hôpital's rule (possibly multiple times) to evaluate the following limits as functions of the real number  $a$ : [Check your results:  $[a, b]$  means that the limit  $L(a) = b$ .]

$$(a) \lim_{x \rightarrow a} \frac{x^2 + (2-a)x - 2a}{x^2 - (a+1)x + a} \quad [2, 4] \qquad (b) \lim_{x \rightarrow 0} \frac{\sinh(x)}{\tanh(ax)} \quad [2, \frac{1}{2}]$$

$$(c) \lim_{x \rightarrow 0} \frac{e^{x^2} - 1}{(e^{ax} - 1)^2} \quad [2, \frac{1}{2}] \qquad (d) \lim_{x \rightarrow 0} \frac{\cosh(ax) + \cos(ax) - 2}{x^4} \quad [2, \frac{4}{3}]$$

(e) Use L'Hôpital's rule to show that for  $\alpha \in \mathbb{R}$  and  $0 < \beta \in \mathbb{R}$  we have

$$\lim_{x \rightarrow 0} (x^\beta \ln^\alpha x) = 0 \quad (\text{with } x > 0),$$

i.e. 'any positive power law beats any power of log'.

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[Total Points for Optional Problems: 16]

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