Sheet 01: Mathematical Foundations

Solution Homework Problem 1: Composition of maps [2]

(a) Since $0^2 = 0$, $(\pm 1)^2 = 1$, $(\pm 2)^2 = 4$, the image of S under A is $T = A(S) = \boxed{\{0, 1, 4\}}$.

(b) Since $\sqrt{0} = 0$, $\sqrt{1} = 1$, $\sqrt{4} = 2$, the image of T under B is $U = B(T) = \boxed{\{0, 1, 2\}}$.

(c) The composite map $C = B \circ A$ is given by $\boxed{C : S \to U, \quad n \mapsto C(n) = \sqrt{n^2} = |n|}$.

(d) A, B and C are all surjective. B is injective and hence also bijective. A and C are not injective, since, e.g., the elements $+2$ and -2 have the same image under A, with $A(\pm 2) = 4$, and similarly $C(\pm 2) = 2$. Therefore, A and C are also not bijective.

Solution Homework Problem 2: The groups of addition modulo 5 and rotations by multiples of 72 deg [3]

(a)

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

The neutral element is 0.
The inverse element of $n \in \mathbb{Z}_q$ is $5 - n$.

(b)

$\cdot$	$r(0)$	$r(72)$	$r(144)$	$r(216)$	$r(288)$
$r(0)$	$r(0)$	$r(72)$	$r(144)$	$r(216)$	$r(288)$
$r(72)$	$r(72)$	$r(144)$	$r(216)$	$r(288)$	$r(0)$
$r(144)$	$r(144)$	$r(216)$	$r(288)$	$r(0)$	$r(72)$
$r(216)$	$r(216)$	$r(288)$	$r(0)$	$r(72)$	$r(144)$
$r(288)$	$r(288)$	$r(0)$	$r(72)$	$r(144)$	$r(216)$

The neutral element is $r(0)$.
The inverse element of $r(\phi)$ is $r(360 - \phi)$.

(c) The groups $(\mathbb{Z}_5, +)$ and $(\mathcal{R}_{72}, \cdot)$ are isomorphic because their group composition tables are identical if we identify the element n of $\mathbb{Z}_5$ with the element $r(72n)$ of $\mathcal{R}_{72}$.

(d) The group $(\mathcal{R}_{360/n}, \cdot)$ of rotations by multiples of $360/n$ deg is isomorphic to the group $(\mathbb{Z}_n, +)$ of integer addition modulo n.

Solution Homework Problem 3: Decomposing permutations into sequences of pair permutations [2]

(a) The permutation $[132]$ is itself a pair permutation, as only the elements 2 and 3 are exchanged, hence its parity is odd.

(b) To obtain $123 \xmapsto{[231]} 231$ via pair permutations, we bring the 2 to the first slot, then the 3 to the second slot: $123 \xmapsto{[213]} 213 \xmapsto{[321]} 231$, thus $[231] = [321] \circ [213]$, with even parity.

Below we proceed similarly: we map the naturally-ordered string into the desired order one pair permutation at a time, moving from front to back:

(c) $1234 \xmapsto{[3214]} 3214 \xmapsto{[1432]} 3412 \quad \Rightarrow \quad [3412] = [1432] \circ [3214]$ even

(d) $1234 \xmapsto{[3214]} 3214 \xmapsto{[1432]} 3412 \xmapsto{[2134]} 3421 \quad \Rightarrow \quad [3421] = [2134] \circ [1432] \circ [3214]$ odd

(e) $12345 \xmapsto{[15342]} 15342 \xmapsto{[13245]} 15243 \xmapsto{[12435]} 15234 \;\Rightarrow [15234] = [12435] \circ [13245] \circ [15342]$ odd

(f) $12345 \xmapsto{[32145]} 32145 \xmapsto{[21345]} 31245 \xmapsto{[15342]} 31542 \;\Rightarrow [31542] = [15342] \circ [21345] \circ [32145]$ odd

Solution Homework Problem 4: Algebraic manipulations with complex numbers [3]

(a) $(z+\mathrm{i})^2 = (x+\mathrm{i}(y+1))^2 = \boxed{x^2-(y+1)^2+\mathrm{i}2x(y+1)}$,

(b)
$$\frac{z}{z+1} = \frac{z}{z+1}\cdot\frac{\bar z+1}{\bar z+1} = \frac{(x+\mathrm{i}y)}{(x+1+\mathrm{i}y)}\cdot\frac{(x+1-\mathrm{i}y)}{(x+1-\mathrm{i}y)}$$
$$= \frac{x(x+1)+y^2+\mathrm{i}(y(x+1)-xy)}{(x+1)^2+y^2} = \boxed{\frac{x(x+1)+y^2+\mathrm{i}y}{(x+1)^2+y^2}},$$

(c)
$$\frac{\bar z}{z-\mathrm{i}} = \frac{\bar z}{z-\mathrm{i}}\cdot\frac{\bar z+\mathrm{i}}{\bar z+\mathrm{i}} = \frac{(x-\mathrm{i}y)}{(x+\mathrm{i}(y-1))}\cdot\frac{(x+\mathrm{i}(1-y))}{(x-\mathrm{i}(y-1))}$$
$$= \frac{x^2+y(1-y)+\mathrm{i}(x(1-y)-yx)}{x^2+(y-1)^2} = \boxed{\frac{x^2+y(1-y)+\mathrm{i}x(1-2y)}{x^2+(y-1)^2}}.$$

Solution Homework Problem 5: Multiplication of complex numbers – geometrical interpretation [2]

$z_1 = \frac{1}{\sqrt8}+\frac{1}{\sqrt8}\mathrm{i} \mapsto (\frac{1}{\sqrt8},\frac{1}{\sqrt8})$ $\quad \rho_1 = \sqrt{\frac18+\frac18} = \frac12$ $\quad \phi_1 = \arctan\big(1\big) = \frac{\pi}{4}$

$z_2 = \sqrt3-\mathrm{i} \mapsto (\sqrt3,-1)$ $\quad \rho_2 = \sqrt{3+1} = 2$ $\quad \phi_2 = \arctan\big(-\frac{1}{\sqrt3}\big) = \frac{11\pi}{6}$

$z_3 = z_1z_2 = (\frac{1}{\sqrt8}+\frac{1}{\sqrt8}\mathrm{i})(\sqrt3-\mathrm{i})$ $\quad \rho_3 = \sqrt{\frac38+\frac18+\frac38+\frac18} = 1$ $\quad \phi_3 = \arctan\big(\frac{\sqrt3-1}{\sqrt3+1}\big) = \frac{\pi}{12}$

$= \sqrt{\frac38}+\sqrt{\frac18}+(\sqrt{\frac38}-\sqrt{\frac18})\mathrm{i} \mapsto (\sqrt{\frac38}+\sqrt{\frac18},\sqrt{\frac38}-\sqrt{\frac18})$

$z_4 = \frac{1}{z_1} = \frac{\sqrt8}{1+\mathrm{i}} = \frac{\sqrt8(1-\mathrm{i})}{(1+\mathrm{i})(1-\mathrm{i})}$ $\quad \rho_4 = \sqrt{2+2} = 2$ $\quad \phi_4 = \arctan\big(-1\big) = \frac{7\pi}{4}$

$= \sqrt2-\sqrt2\mathrm{i} \mapsto (\sqrt2,-\sqrt2)$

$z_5 = \bar{z}_1 = \frac{1}{\sqrt{8}} - \frac{1}{\sqrt{8}}\mathrm{i} \mapsto (\frac{1}{\sqrt{8}}, -\frac{1}{\sqrt{8}})$ $\qquad \rho_5 = \sqrt{\frac{1}{8}+\frac{1}{8}} = \frac{1}{2}$ $\qquad \phi_5 = \arctan\big(-1\big) = \frac{7\pi}{4}$

As expected, we find:

$\rho_3 = \rho_1\rho_2$

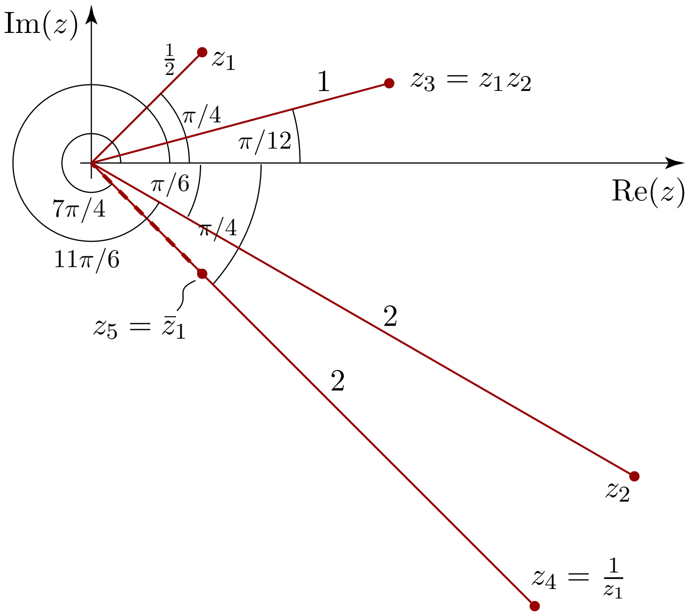
$\phi_3 = \phi_1 + \phi_2$

$\rho_4 = 1/\rho_1$

$\phi_4 = -\phi_1$

$\rho_5 = \rho_1$

$\phi_5 = -\phi_1$



Solution Homework Problem 6: Differentiation of hyperbolic functions [2]

(a) $\cosh^2 x - \sinh^2 x = \frac{1}{4}(\mathrm{e}^x + \mathrm{e}^{-x})^2 - \frac{1}{4}(\mathrm{e}^x - \mathrm{e}^{-x})^2$

$$= \tfrac{1}{4}(\mathrm{e}^{2x} + 2 + \mathrm{e}^{-2x}) - \tfrac{1}{4}(\mathrm{e}^{2x} - 2 + \mathrm{e}^{-2x}) = \boxed{1}. \checkmark$$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\sinh x = \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x}) = \frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x}) = \boxed{\cosh x}. \checkmark$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}\cosh x = \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x}) = \frac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x}) = \boxed{\sinh x}. \checkmark$

(d) $\frac{\mathrm{d}}{\mathrm{d}x}\tanh x = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\sinh x}{\cosh x} = \frac{\cosh x}{\cosh x} - \frac{\sinh^2 x}{\cosh^2 x} = \boxed{1-\tanh^2 x}, \checkmark$

$$= \tfrac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \tfrac{1}{\cosh^2 x} = \boxed{\operatorname{sech}^2 x}. \checkmark$$

(e) $\frac{\mathrm{d}}{\mathrm{d}x}\coth x = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\cosh x}{\sinh x} = \frac{\sinh x}{\sinh x} - \frac{\cosh^2 x}{\sinh^2 x} = \boxed{1-\coth^2 x}, \checkmark$

$$= \tfrac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = -\tfrac{1}{\sinh^2 x} = \boxed{-\operatorname{csch}^2 x}. \checkmark$$

Solution Homework Problem 7: Differentiation of powers, exponentials, logarithms [2]

(a) $f'(x) = \frac{2}{3\sqrt[3]{x}}$

(b) $f'(x) = \dfrac{1}{(x^2+1)^{1/2}} - \dfrac{1}{2}\dfrac{x\cdot 2x}{(x^2+1)^{3/2}} = \dfrac{1}{(x^2+1)^{3/2}}$

(c) $f'(x) = 2x\mathrm{e}^{1-x^2}$

(d) $f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{\ln 2^{x^2}} = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{x^2\ln 2} = \mathrm{e}^{x^2\ln 2}2x\ln 2 = 2^{x^2}2x\ln 2$

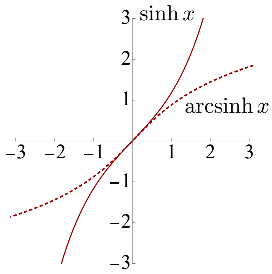
(e) $f'(x) = \dfrac{1}{\sqrt{\ln x}}\dfrac{1}{x^2} - 2\dfrac{\sqrt{\ln x}}{x^2}$

(f) $f'(x) = \dfrac{1}{\sqrt{x^2+1}}\dfrac{1}{2}\dfrac{2x}{\sqrt{x^2+1}} = \dfrac{x}{x^2+1}$

Solution Homework Problem 8: Differentiation of inverse hyperbolic functions [2]

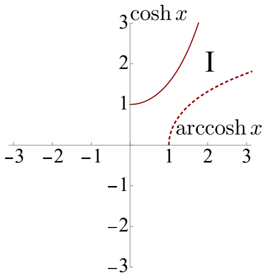
The hyperbolic functions $f = \sinh$ and $\tanh$ are monotonic, hence the same is true for their inverses, $f^{-1} = \operatorname{arcsinh}$ and $\operatorname{arctanh}$. However $\cosh(x)$ is non-monotonic, with positive/negative slope for $x \gtrless 0$, hence its inverse, $\operatorname{arccosh}$, has two branches, which we consider separately. For each case we compute the derivative of f^{-1} using $(f^{-1})'(x) = \frac{1}{f'(y)|_{y=f^{-1}(x)}}$.

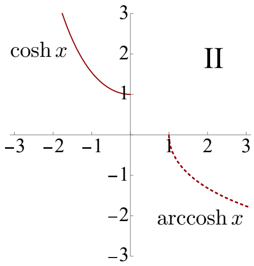
(a) $\operatorname{arcsinh}$ is the inverse of $\sinh$, with $\sinh(\operatorname{arcsinh} x) = x$. The slope of $\sinh$, given by $\sinh' x = \cosh x$, is positive for all $x \in \mathbb{R}$. Hence $\sinh\colon \mathbb{R} \to \mathbb{R}$ is monotonic, and so is its inverse, $\operatorname{arcsinh}\colon \mathbb{R} \to \mathbb{R}$.



$$\begin{aligned}\operatorname{arcsinh}' x &= \frac{1}{\sinh'(y)|_{y=\operatorname{arcsinh} x}} = \frac{1}{\cosh(\operatorname{arcsinh} x)} \\ &= \frac{1}{\sqrt{1+\sinh^2(\operatorname{arcsinh} x)}} = \boxed{\frac{1}{\sqrt{1+x^2}}}.\end{aligned}$$

(b) $\operatorname{arccosh}$ is the inverse of $\cosh$, with $\cosh(\operatorname{arccosh} x) = x$.
We consider the two branches of $\operatorname{arccosh}$, with slopes of opposite sign, separately.
I: The function $\cosh\colon (0, \infty) \to (1, \infty)$ has positive slope, $\cosh' x = \sinh x$, and inverse $\operatorname{arccosh}\colon (1, \infty) \to (0, \infty)$.
II: The function $\cosh\colon (-\infty, 0) \to (\infty, 1)$ has negative slope, $\cosh' x = \sinh x$, and inverse $\operatorname{arccosh}\colon (1, \infty) \to (0, -\infty)$.
Using upper/lower signs for branch I/II, we obtain

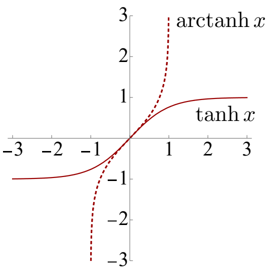




$$\begin{aligned}\operatorname{arccosh}' x &= \frac{1}{\cosh'(y)|_{y=\operatorname{arccosh} x}} = \frac{1}{\sinh(\operatorname{arccosh} x)} \\ &= \frac{\pm 1}{\sqrt{\cosh^2(\operatorname{arccosh} x) - 1}} = \boxed{\frac{\pm 1}{\sqrt{x^2-1}}}.\end{aligned}$$

Unless stated otherwise, the notation $\operatorname{arccosh}$ refers to branch I.

(c) $\operatorname{arctanh}$ is the inverse of $\tanh$, with $\tanh(\operatorname{arctanh} x) = x$. The slope of $\tanh$, given by $\tanh' x = \operatorname{sech}^2 x$, is positive for all $x \in \mathbb{R}$. Hence $\tanh\colon \mathbb{R} \to (-1, 1)$ is monotonic, and so is its inverse, $\operatorname{arctanh}\colon (-1, 1) \to \mathbb{R}$.



$$\begin{aligned}\operatorname{arctanh}' x &= \frac{1}{\tanh'(y)|_{y=\operatorname{arctanh} x}} = \frac{1}{\operatorname{sech}^2(\operatorname{arctanh} x)} \\ &= \frac{1}{1-\tanh^2(\operatorname{arctanh} x)} = \boxed{\frac{1}{1-x^2}}.\end{aligned}$$

Solution Homework Problem 9: Integration by parts [4]

(a) $$\begin{aligned}I(z) &= \int_0^z \mathrm{d}x\, \overset{u}{x}\, \overset{v'}{\sin(2x)} = \Big[\overset{u}{x}\, \overset{v}{\left(-\tfrac{1}{2}\cos(2x)\right)}\Big]_0^z - \int_0^z \mathrm{d}x\, \overset{u'}{1} \cdot \overset{v}{\left(-\tfrac{1}{2}\cos(2x)\right)} \\ &= \boxed{-\tfrac{1}{2}z\cos(2z) + \tfrac{1}{4}\sin(2z)}\end{aligned}$$

$$I'(z) = -\tfrac{1}{2}\left[\cos(2z) - z2\sin(2z)\right] + \tfrac{1}{4}2\cos(2z) \overset{\checkmark}{=} z\sin(2z) \qquad I(\tfrac{\pi}{2}) \overset{\checkmark}{=} \tfrac{\pi}{4}$$

(b) $$I(z) = \int_0^z \mathrm{d}x\, \overset{u}{x^2}\, \overset{v'}{\cos(2x)} = \left[\overset{u}{x^2}\, \overset{v}{\tfrac{1}{2}\sin(2x)}\right]_0^z - \int_0^z \mathrm{d}x\, \overset{u'}{2x}\, \overset{v}{\tfrac{1}{2}\sin(2x)}\ ,$$ use (a):

$$\overset{\text{(a)}}{=} \boxed{\tfrac{1}{2}z^2\sin(2z) + \tfrac{1}{2}z\cos(2z) - \tfrac{1}{4}\sin(2z)} \qquad I(\tfrac{\pi}{2}) \overset{\checkmark}{=} -\tfrac{\pi}{4}$$

$$I'(z) = z\sin(2z) + z^2\cos(2z) + \tfrac{1}{2}\cos(2z) - z\sin(2z) - \tfrac{1}{2}\cos(2z) \overset{\checkmark}{=} z^2\cos(2z)$$

(c) $$I(z) = \int_0^z \mathrm{d}x\, \overset{u}{(\ln x)}\, \overset{v'}{x} = \left[\overset{u}{(\ln x)}\, \overset{v}{\tfrac{1}{2}x^2}\right]_0^z - \int_0^z \mathrm{d}x\, \overset{u'}{\tfrac{1}{x}}\, \overset{v}{\tfrac{1}{2}x^2} = \boxed{(\ln z)\tfrac{1}{2}z^2 - \tfrac{1}{4}z^2}$$

$$I'(z) = \tfrac{1}{z}\tfrac{1}{2}z^2 + (\ln z)z - \tfrac{1}{2}z \overset{\checkmark}{=} (\ln z)z \qquad I(1) \overset{\checkmark}{=} -\tfrac{1}{4}$$

(d) $$I(z) = \int_0^z \mathrm{d}x\, \overset{u}{(\ln x)}\, \overset{v'}{x^n} = \left[\overset{u}{(\ln x)}\, \overset{v}{\tfrac{1}{n+1}x^{n+1}}\right]_0^z - \int_0^z \mathrm{d}x\, \overset{u'}{\tfrac{1}{x}}\, \overset{v}{\tfrac{1}{n+1}x^{n+1}}$$

$$= \boxed{(\ln z)\tfrac{1}{n+1}z^{n+1} - \tfrac{1}{(n+1)^2}z^{n+1}}$$ [for $n > -1$]

To evaluate $\left[\ln(x)x^{n+1}\right]_{x=0}$, we set $m = n+1 > 0$ and used the rule of L'Hôpital (see sheet 01, optional problems 3,4):

$$\left[\ln(x)x^m\right]_{x=0} = \lim_{x\to 0}\frac{\ln x}{x^{-m}} = \lim_{x\to 0}\frac{\frac{\mathrm{d}}{\mathrm{d}x}\ln x}{\frac{\mathrm{d}}{\mathrm{d}x}x^{-m}} = \lim_{x\to 0}\frac{x^{-1}}{-mx^{-(m+1)}} = -\lim_{x\to 0}\left[\frac{x^m}{m}\right] \overset{m>0}{=} \boxed{0}\,.$$

The divergence of $\ln(x)$ for $x \to 0$ is so slow that any positive power of x suppresses it.

$$I'(z) = \tfrac{1}{z}\tfrac{1}{n+1}z^{n+1} + (\ln z)z^n - \tfrac{1}{n+1}z^n \overset{\checkmark}{=} (\ln z)z^n \qquad I(1) \overset{\checkmark}{=} \tfrac{-1}{(n+1)^2}$$

(e) $$I(z) = \int_0^z \mathrm{d}x\, \overset{u}{\cos x}\, \overset{v'}{\cos x} = \left[\overset{u}{\cos x}\, \overset{v}{\sin x}\right]_0^z - \int_0^z \mathrm{d}x\, \underbrace{(\overset{u'}{-\sin x}\, \overset{v}{\sin x})}_{\cos^2 x - 1}$$

$$= \cos z\sin z - I(z) + \int_0^z \mathrm{d}x\ 1\ ,$$ solve for $I(z)$:

$$I(z) = \boxed{\tfrac{1}{2}(\cos z\sin z + z)}$$

$$I'(z) = \tfrac{1}{2}(-\sin^2 z + \cos^2 z + 1) \overset{\checkmark}{=} \cos^2 z \qquad I(\pi) \overset{\checkmark}{=} \tfrac{\pi}{2}$$

(f) $$I(z) = \int_0^z \mathrm{d}x\, \overset{u}{\cos^3 x}\, \overset{v'}{\cos x} = \left[\overset{u}{\cos^3 x}\, \overset{v}{\sin x}\right]_0^z - \int_0^z \mathrm{d}x\, \overset{u'}{\left(-3\cos^2 x\sin x\right)}\, \overset{v}{\sin x}$$

with $\underbrace{\sin x\,\sin x}_{1-\cos^2 x}$

$$= \cos^3 z\sin z - 3\left[I(z) - \int_0^z \mathrm{d}x\ \cos^2 x\right]\ ,$$ solve for $I(z)$, use (e):

$$I(z) \overset{\text{(e)}}{=} \boxed{\tfrac{1}{4}\left[\cos^3 z\sin z + \tfrac{3}{2}(\cos z\sin z + z)\right]} \qquad I(\pi) \overset{\checkmark}{=} \tfrac{3\pi}{8}$$

$$I'(z) = \tfrac{1}{4}\Big[-3\cos^2 z \underbrace{\sin^2 z}_{1-\cos^2 z} + \cos^4 z + \tfrac{3}{2}(-\sin^2 z + \cos^2 z + 1)\Big] \overset{\checkmark}{=} \cos^4 z$$

Solution Homework Problem 10: Integration by substitution [3]

(a)
$$\begin{aligned}
I(z) &= \int_0^z \mathrm{d}x\; x^2\,\sqrt{x^3+1} \qquad [y(x) = x^3+1,\ \mathrm{d}y = 3x^2\,\mathrm{d}x] \\
&= \tfrac{1}{3}\int_{y(0)}^{y(z)} \mathrm{d}y\;\sqrt{y} = \tfrac{2}{9}y^{3/2}\Big|_1^{z^3+1} = \boxed{\tfrac{2}{9}\left[(z^3+1)^{3/2}-1\right]} \\
I'(z) &= \tfrac{1}{3}(z^3+1)^{1/2}\,\tfrac{\mathrm{d}}{\mathrm{d}z}z^3 \overset{\checkmark}{=} \sqrt{z^3+1}\;z^2 \qquad\qquad I(2) \overset{\checkmark}{=} \tfrac{52}{9}
\end{aligned}$$

(b)
$$\begin{aligned}
I(z) &= \int_0^z \mathrm{d}x\;\sin x\;\mathrm{e}^{\cos x} \qquad [y(x) = \cos x,\ \mathrm{d}y = -\sin x\,\mathrm{d}x] \\
&= -\int_{y(0)}^{y(z)} \mathrm{d}y\;\mathrm{e}^y = -\mathrm{e}^y\Big|_1^{\cos z} = \boxed{\mathrm{e}-\mathrm{e}^{\cos z}} \\
I'(z) &= -\mathrm{e}^{\cos z}\tfrac{\mathrm{d}}{\mathrm{d}z}\cos z \overset{\checkmark}{=} \mathrm{e}^{\cos z}\sin z \qquad\qquad I(\tfrac{\pi}{3}) \overset{\checkmark}{=} \mathrm{e}-\sqrt{\mathrm{e}}
\end{aligned}$$

(c)
$$\begin{aligned}
I(z) &= \int_0^z \mathrm{d}x\cos^3 x = \int_0^z \mathrm{d}x\cos x\big[1-\sin^2 x\big] \qquad [y(x) = \sin x,\ \mathrm{d}y = \cos x\,\mathrm{d}x] \\
&= \int_{y(0)}^{y(z)} \mathrm{d}y\;(1-y^2) = (y-\tfrac{1}{3}y^3)\Big|_0^{\sin z} = \boxed{\sin z - \tfrac{1}{3}\sin^3 z} \\
I'(z) &= \cos z - \sin^2 z\cos z = \cos z(1-\sin^2 z) \overset{\checkmark}{=} \cos^3 z \qquad\qquad I(\tfrac{\pi}{4}) \overset{\checkmark}{=} \tfrac{5}{6\sqrt{2}}
\end{aligned}$$

(d)
$$\begin{aligned}
I(z) &= \int_0^z \mathrm{d}x\sinh^3 x = \int_0^z \mathrm{d}x\sinh x\big[\cosh^2 x - 1\big] \quad [y(x) = \cosh x,\ \mathrm{d}y = \sinh x\mathrm{d}x] \\
&= \int_{y(0)}^{y(z)} \mathrm{d}y\;(y^2-1) = (\tfrac{1}{3}y^3-y)\Big|_1^{\cosh z} = \boxed{\tfrac{1}{3}\cosh^3 z - \cosh z + \tfrac{2}{3}} \\
I'(z) &= \cosh^2 z\sinh z - \sinh z = (\cosh^2 z - 1)\sinh z \overset{\checkmark}{=} \sinh^3 z \qquad I(\ln 3) \overset{\checkmark}{=} \tfrac{44}{81}
\end{aligned}$$

(e)
$$\begin{aligned}
I(z) &= \int_0^z \mathrm{d}x\;\sin\sqrt{\pi x}\tfrac{1}{\sqrt{x}} \qquad \Big[y(x) = \sqrt{\pi x},\ \mathrm{d}y = \tfrac{1}{2}\sqrt{\pi/x}\;\mathrm{d}x\Big] \\
&= \tfrac{2}{\sqrt{\pi}}\int_{y(0)}^{y(z)} \mathrm{d}y\;\sin y = -\tfrac{2}{\sqrt{\pi}}\cos y\Big|_0^{\sqrt{\pi z}} = \boxed{\tfrac{2}{\sqrt{\pi}}\left[1-\cos\sqrt{\pi z}\right]} \\
I'(z) &= \tfrac{2}{\sqrt{\pi}}\sin\sqrt{\pi z}\;\tfrac{\mathrm{d}}{\mathrm{d}z}\sqrt{\pi z} \overset{\checkmark}{=} \sin\sqrt{\pi z}\tfrac{1}{\sqrt{z}} \qquad\qquad I(\tfrac{\pi}{9}) \overset{\checkmark}{=} \tfrac{1}{\sqrt{\pi}}
\end{aligned}$$

(f)
$$\begin{aligned}
I(z) &= \int_0^z \mathrm{d}x\;\sqrt{x}\;\mathrm{e}^{\sqrt{x^3}} \qquad \big[y(x) = x^{3/2},\ \mathrm{d}y = \tfrac{3}{2}x^{1/2}\;\mathrm{d}x\big] \\
&= \tfrac{2}{3}\int_{y(0)}^{y(z)} \mathrm{d}y\;\mathrm{e}^y = \tfrac{2}{3}\mathrm{e}^y\Big|_0^{z^{3/2}} = \boxed{\tfrac{2}{3}\left[\mathrm{e}^{z^{3/2}}-1\right]} \\
I'(z) &= \tfrac{2}{3}\mathrm{e}^{z^{3/2}}\tfrac{\mathrm{d}}{\mathrm{d}z}z^{3/2} \overset{\checkmark}{=} \mathrm{e}^{z^{3/2}}z^{1/2} \qquad\qquad I\left((\ln 4)^{2/3}\right) \overset{\checkmark}{=} 2
\end{aligned}$$

[Total Points for Homework Problems: 25]
