Sheet 01: Mathematical Foundations

Solution Example Problem 1: Composition of maps [2]

(a) Since A maps $\mathbb{Z}$ to $\mathbb{Z}$ and B maps $\mathbb{Z}$ to $\mathbb{N}_0$, it follows that $C = B \circ A$ maps $\mathbb{Z}$ to $\mathbb{N}_0$. The image of n is $C(n) = B(A(n)) = B(n+1) = |n+1|$. To summarize:

$$\boxed{C : \mathbb{Z} \to \mathbb{N}_0, \quad n \mapsto C(n) = |n+1|.}$$

(b) A, B and C are all surjective. A is also injective and bijective. B is not injective, because any positive $n \in \mathbb{N}_0$ is the image of *two* points in $\mathbb{Z}$, $B(n) = B(-n) = n$. Consequently, B is not bijective either. It follows that C, too, is not injective and thus not bijective.

Solution Example Problem 2: The abelian group $\mathbb{Z}_2$ [3]

(a) The composition table implies the following properties:

+	0	1
0	0	1
1	1	0

(i) Closure: the result of any possible addition is listed in the table and belongs to the set $\{0, 1\}$. ✓

(i) Associativity:
$(1 \boldsymbol{+} 0) \boldsymbol{+} 0 = 1 \boldsymbol{+} 0 = 1 \stackrel{?}{=} 1 \boldsymbol{+} (0 \boldsymbol{+} 0) = 1 \boldsymbol{+} 0 = 1$ ✓
$(0 \boldsymbol{+} 1) \boldsymbol{+} 0 = 1 \boldsymbol{+} 0 = 1 \stackrel{?}{=} 0 \boldsymbol{+} (1 \boldsymbol{+} 0) = 0 \boldsymbol{+} 1 = 1$ ✓
$(1 \boldsymbol{+} 1) \boldsymbol{+} 0 = 0 \boldsymbol{+} 0 = 0 \stackrel{?}{=} 1 \boldsymbol{+} (1 \boldsymbol{+} 0) = 1 \boldsymbol{+} 1 = 0$ ✓
$(1 \boldsymbol{+} 0) \boldsymbol{+} 1 = 1 \boldsymbol{+} 1 = 0 \stackrel{?}{=} 1 \boldsymbol{+} (0 \boldsymbol{+} 1) = 1 \boldsymbol{+} 1 = 0$ ✓
$(0 \boldsymbol{+} 1) \boldsymbol{+} 1 = 1 \boldsymbol{+} 1 = 0 \stackrel{?}{=} 0 \boldsymbol{+} (1 \boldsymbol{+} 1) = 0 \boldsymbol{+} 0 = 0$ ✓
$(0 \boldsymbol{+} 0) \boldsymbol{+} 1 = 0 \boldsymbol{+} 1 = 1 \stackrel{?}{=} 0 \boldsymbol{+} (0 \boldsymbol{+} 1) = 0 \boldsymbol{+} 1 = 1$ ✓

(ii) The neutral element is 0, since adding it yields no change: $0 \boldsymbol{+} 0 = 0$, $0 \boldsymbol{+} 1 = 1$.

(iii) For every element in the group, there is exactly one inverse, since every row of the table contains exactly one 0.

(iv) The group is abelian since the table is symmetric with respect to the diagonal.

(b) The group $(\{+1, -1\}, \bullet)$, with standard multiplication as group operation, is isomorphic to $\mathbb{Z}_2$, since their composition tables have the same structure if we identify $+1$ with 0 and -1 with 1.

$\bullet$	$+1$	-1
$+1$	$+1$	-1
-1	-1	$+1$

Solution Example Problem 3: Permutation groups [4]

(a) The entries of the composition table can found by evaluating the image of 123 under P followed by P'. For example $123 \xmapsto{[213]} 213 \xmapsto{[321]} 231$, hence $[321] \circ [213] = [231]$.

$P' \circ P$	$[123]$	$[231]$	$[312]$	$[213]$	$[321]$	$[132]$
$[123]$	$[123]$	$[231]$	$[312]$	$[213]$	$[321]$	$[132]$
$[231]$	$[231]$	$[312]$	$[123]$	$[321]$	$[132]$	$[213]$
$[312]$	$[312]$	$[123]$	$[231]$	$[132]$	$[213]$	$[321]$
$[213]$	$[213]$	$[132]$	$[321]$	$[123]$	$[312]$	$[231]$
$[321]$	$[321]$	$[213]$	$[132]$	$[231]$	$[123]$	$[312]$
$[132]$	$[132]$	$[321]$	$[213]$	$[312]$	$[231]$	$[123]$

(b) The neutral element is the permutation that 'does nothing', $[123]$. Each element has a unique inverse, since every row and column contains the neutral element exactly once.

(c) The composition table is not symmetric, $P' \circ P \neq P \circ P'$, hence S_3 is *not* an abelian group. For example, $[312] \circ [213] = [132]$, whereas $[213] \circ [312] = [321]$.

Solution Example Problem 4: Algebraic manipulations with complex numbers [4]

(a) $$z + \bar{z} = x + \mathrm{i}y + x - \mathrm{i}y = \boxed{2x} = 2\mathrm{Re}(z)\,,$$

(b) $$z - \bar{z} = x + \mathrm{i}y - (x - \mathrm{i}y) = \boxed{\mathrm{i}2y} = \mathrm{i}2\mathrm{Im}(z)\,,$$

(c) $$z \cdot \bar{z} = (x + \mathrm{i}y)(x - \mathrm{i}y) = \boxed{x^2 + y^2}\,,$$

(d) $$\frac{z}{\bar{z}} \overset{\text{(c)}}{=} \frac{z \cdot z}{\bar{z} \cdot z} = \frac{(x + \mathrm{i}y)^2}{x^2 + y^2} = \boxed{\frac{x^2 - y^2}{x^2 + y^2} + \mathrm{i}\frac{2xy}{x^2 + y^2}}\,,$$

(e) $$\frac{1}{z} + \frac{1}{\bar{z}} = \frac{\bar{z} + z}{z \cdot \bar{z}} \overset{\text{(a),(c)}}{=} \boxed{\frac{2x}{x^2 + y^2}}\,,$$

(f) $$\frac{1}{z} - \frac{1}{\bar{z}} = \frac{\bar{z} - z}{z \cdot \bar{z}} \overset{\text{(b),(c)}}{=} \boxed{\mathrm{i}\frac{(-2y)}{x^2 + y^2}}\,,$$

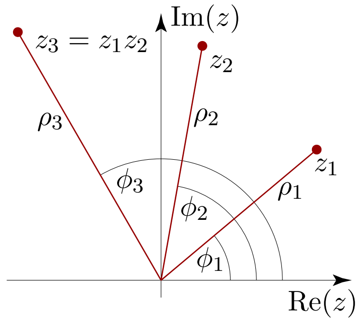
(g) $$z^2 + z = (x + \mathrm{i}y)^2 + (x + \mathrm{i}y) = \boxed{(x^2 - y^2 + x) + \mathrm{i}(2xy + y)}\,,$$

(h) $$z^3 = (x + \mathrm{i}y)^3 = (x^3 + 3x^2\mathrm{i}y + 3x(\mathrm{i}y)^2 + (\mathrm{i}y)^3 = \boxed{(x^3 - 3xy^2) + \mathrm{i}(3x^2y - y^3)}\,.$$

Solution Example Problem 5: Multiplication of complex numbers – geometrical interpretation [4]

(a) With $z_j = (\rho_j \cos\phi_j, \rho_j \sin\phi_j)$ and the given trigonometric identities, we have

$$\begin{aligned} z_3 = z_1 z_2 &= \rho_1(\cos\phi_1 + \mathrm{i}\sin\phi_1)\rho_2(\cos\phi_2 + \mathrm{i}\sin\phi_2) \\ &= \rho_1\rho_2\left[(\cos\phi_1\cos\phi_2 - \sin\phi_1\sin\phi_2)\right. \\ &\qquad \left. + \mathrm{i}\,(\sin\phi_1\cos\phi_2 + \cos\phi_1\sin\phi_2)\right] \\ &= \rho_1\rho_2\left[\cos(\phi_1+\phi_2) + \mathrm{i}\sin(\phi_1+\phi_2)\right] \\ &\equiv \rho_3\left[\cos\phi_3 + \mathrm{i}\sin\phi_3\right] \end{aligned}$$

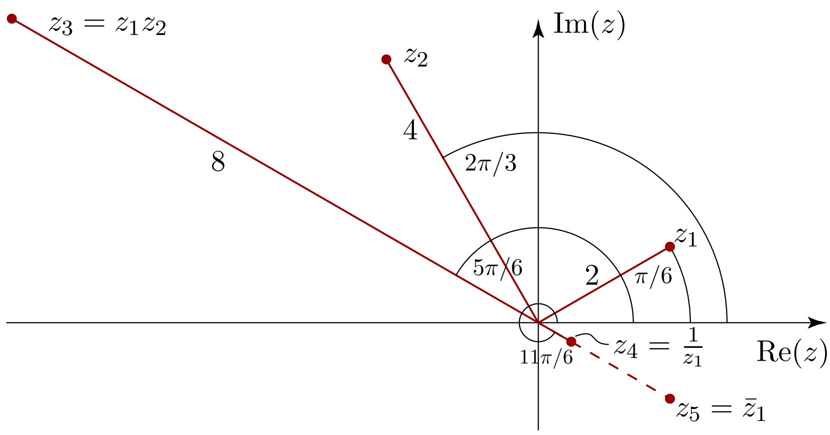


We read off: $\rho_3 = \rho_1\rho_2$, $\phi_3 = (\phi_1+\phi_2)\,\mathrm{mod}(2\pi)$. ✓

(b) The complex number $z = x + \mathrm{i}y$ is represented in the complex plane by the Cartesian coordinates $z \mapsto (x, y)$, or the polar coordinates $\rho = |z| = \sqrt{x^2+y^2}$, $\phi = \arg(z) = \arctan\left(\frac{y}{x}\right)$. The latter formula determines ϕ only modulo π; to uniquely fix $\phi \in [0, 2\pi)$, we identify the quadrant containing the point (x, y).

$$\begin{aligned}
&z_1 = \sqrt{3}+\mathrm{i} \mapsto (\sqrt{3}, 1) && \rho_1 = \sqrt{3+1} = 2 && \phi_1 = \arctan\left(\tfrac{1}{\sqrt{3}}\right) = \tfrac{\pi}{6} \\
&z_2 = -2+2\sqrt{3}\mathrm{i} \mapsto (-2, 2\sqrt{3}) && \rho_2 = \sqrt{12+4} = 4 && \phi_2 = \arctan\left(\tfrac{-2\sqrt{3}}{2}\right) = \tfrac{2\pi}{3} \\
&z_3 = z_1 z_2 = (\sqrt{3}+\mathrm{i})(-2+2\sqrt{3}\mathrm{i}) && \rho_3 = \sqrt{16\cdot 3+16} = 8 && \phi_3 = \arctan\left(\tfrac{4}{-4\sqrt{3}}\right) = \tfrac{5\pi}{6} \\
&\quad = -4\sqrt{3}+4\mathrm{i} \mapsto (-4\sqrt{3}, 4) && && \\
&z_4 = \tfrac{1}{z_1} = \tfrac{1}{\sqrt{3}+\mathrm{i}} = \tfrac{(\sqrt{3}-\mathrm{i})}{(\sqrt{3}+\mathrm{i})(\sqrt{3}-\mathrm{i})} && \rho_4 = \tfrac{1}{4}\sqrt{3+1} = \tfrac{1}{2} && \phi_4 = \arctan\left(\tfrac{-1/4}{\sqrt{3}/4}\right) = \tfrac{11\pi}{6} \\
&\quad = \tfrac{\sqrt{3}}{4} - \tfrac{1}{4}\mathrm{i} \mapsto \left(\tfrac{\sqrt{3}}{4}, -\tfrac{1}{4}\right) && && \\
&z_5 = \bar{z}_1 = \sqrt{3} - \mathrm{i} \mapsto (\sqrt{3}, -1) && \rho_5 = \sqrt{3+1} = 2 && \phi_5 = \arctan\left(\tfrac{-1}{\sqrt{3}}\right) = \tfrac{11\pi}{6}
\end{aligned}$$

As expected, we find:
$\rho_3 = \rho_1\rho_2$
$\phi_3 = \phi_1+\phi_2$
$\rho_4 = 1/\rho_1$
$\phi_4 = -\phi_1 \mathrm{mod}(2\pi)$
$\rho_5 = \rho_1$
$\phi_5 = -\phi_1 \mathrm{mod}(2\pi)$



Solution Example Problem 6: Differentiation of trigonometric functions [1]

Using $\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x$, $\frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x$, and $\sin^2 x + \cos^2 x = 1$, we readily find

(a) $$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}x}\tan x = \tfrac{\mathrm{d}}{\mathrm{d}x}\tfrac{\sin x}{\cos x} &= \tfrac{\cos x}{\cos x} + \tfrac{\sin^2 x}{\cos^2 x} = \boxed{1 + \tan^2 x}, ✓ \\ &= \tfrac{\cos^2 x + \sin^2 x}{\cos^2 x} = \tfrac{1}{\cos^2 x} = \boxed{\sec^2 x}. ✓ \end{aligned}$$

(b) $$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}x}\cot x = \tfrac{\mathrm{d}}{\mathrm{d}x}\tfrac{\cos x}{\sin x} &= -\tfrac{\sin x}{\sin x} - \tfrac{\cos^2 x}{\sin^2 x} = \boxed{-1 - \cot^2 x}, ✓ \\ &= \tfrac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\tfrac{1}{\sin^2 x} = \boxed{-\csc^2 x}. ✓ \end{aligned}$$

Solution Example Problem 7: Differentiation of powers, exponentials, logarithms [2]

(a) $f'(x) = \frac{1}{2\sqrt{2x^3}}$

(b) $f'(x) = \frac{1}{2}\frac{1}{x^{1/2}(x+1)^{1/2}} - \frac{1}{2}\frac{x^{1/2}}{(x+1)^{3/2}} = \frac{1}{2}\frac{1}{x^{1/2}(x+1)^{3/2}}$

(c) $f'(x) = \mathrm{e}^x(2x-1)$

(d) $f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{\ln 3^x} = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{x\ln 3} = \mathrm{e}^{x\ln 3}\ln 3 = 3^x \ln 3$

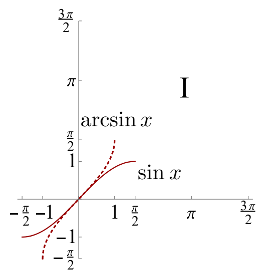
(e) $f'(x) = \ln x + \frac{x}{x} = \ln x + 1$

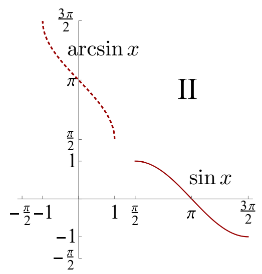
(f) $f'(x) = \ln(9x^2) + x\frac{1}{9x^2}18x = \ln(9x^2) + 2$

Solution Example Problem 8: Differentiation of inverse trigonometric functions [4]

The trigonometric functions $f = \sin$, $\cos$ and $\tan$ are all periodic, hence their inverses, $f^{-1} = \arcsin$, $\arccos$ and $\arctan$, each have infinitely many branches, one for each x-domain of f on which a bijection can be defined. On any given branch, the slope of f^{-1} has the same sign as the slope of f. We consider representative examples of such branches, and for each case compute the derivative of f^{-1} using $(f^{-1})'(x) = \frac{1}{f'(y)|_{y=f^{-1}(x)}}$.

(a) $\arcsin$ is the inverse function of $\sin$, with $\sin(\arcsin x) = x$. We consider two branches of $\arcsin x$, with slopes of opposite sign.
I: The function $\sin\colon (-\frac{1}{2}\pi, \frac{1}{2}\pi) \to (-1, 1)$ has positive slope, $\sin' x = \cos x$, and inverse $\arcsin\colon (-1,1) \to (-\frac{1}{2}\pi, \frac{1}{2}\pi)$.
II: The function $\sin\colon (\frac{1}{2}\pi, \frac{3}{2}\pi) \to (1, -1)$ has negative slope, $\sin' x = \cos x$, and inverse $\arcsin\colon (-1,1) \to (\frac{3}{2}\pi, \frac{1}{2}\pi)$.
Using upper/lower signs for branch I/II, we obtain

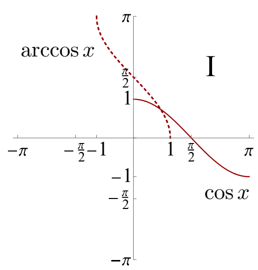


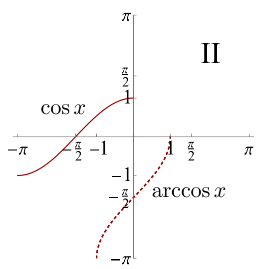


$$\arcsin' x = \frac{1}{\sin'(y)|_{y=\arcsin x}} = \frac{1}{\cos(\arcsin x)}$$
$$= \frac{\pm 1}{\sqrt{1-\sin^2(\arcsin x)}} = \boxed{\frac{\pm 1}{\sqrt{1-x^2}}}.$$

Unless stated otherwise, the notation $\arcsin$ refers to branch I.

(b) $\arccos$ is the inverse function of $\cos$, with $\cos(\arccos x) = x$. We consider two branches of $\arccos$, with slopes of opposite sign.
I: The function $\cos\colon (0,\pi) \to (1,-1)$ has negative slope, $\cos' x = -\sin x$, and inverse $\arccos\colon (-1,1) \to (\pi, 0)$.
II: The function $\cos x\colon (-\pi, 0) \to (-1,1)$ has positive slope, $\cos' x = -\sin x$, and inverse $\arccos\colon (-1,1) \to (-\pi, 0)$.
Using upper/lower signs for branch I/II, we obtain

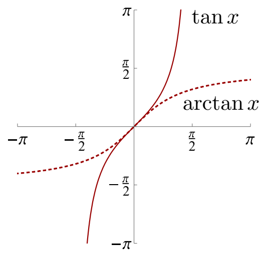




$$\arccos' x = \frac{1}{\cos'(y)|_{y=\arccos x}} = \frac{-1}{\sin(\arccos x)}$$
$$= \frac{\mp 1}{\sqrt{1-\cos^2(\arccos x)}} = \boxed{\frac{\mp 1}{\sqrt{1-x^2}}}.$$

Unless stated otherwise, the notation $\arccos$ refers to branch I.

(c) arctan is the inverse function of tan, with $\tan(\arctan x) = x$. The slope of tan, given by $\tan' x = \sec^2 x$, is positive for every branch. We consider only the branch centered on zero, $\tan\colon (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$, with inverse $\arctan\colon \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$:



$$\arctan' x = \frac{1}{\tan'(y)|_{y=\arctan x}} = \frac{1}{\sec^2(\arctan x)}$$
$$= \frac{1}{1+\tan^2(\arctan x)} = \boxed{\frac{1}{1+x^2}}.$$

Solution Example Problem 9: Integration by parts [6]

(a) $$I(z) = \int_0^z \mathrm{d}x\, \overset{u}{x}\, \overset{v'}{\mathrm{e}^{2x}} = \Big[\overset{u}{x}\, \overset{v}{\tfrac12 \mathrm{e}^{2x}}\Big]_0^z - \int_0^z \mathrm{d}x\, \overset{u'}{1} \cdot \overset{v}{\tfrac12 \mathrm{e}^{2x}} = \boxed{\tfrac12 z\mathrm{e}^{2z} - \tfrac14\left[\mathrm{e}^{2z} - 1\right]}$$

$$I'(z) = \left[\tfrac12(1+2z) - \tfrac14 2\right]\mathrm{e}^{2z} \overset{\checkmark}{=} z\mathrm{e}^{2z} \qquad\qquad I(\tfrac12) \overset{\checkmark}{=} \tfrac14$$

Note the cancellation pattern: $I' = u'v + uv' - u'v = uv'$. [Similarly for (c,d).]

(b) $$I(z) = \int_0^z \mathrm{d}x\, \overset{u}{x^2}\, \overset{v'}{\mathrm{e}^{2x}} = \Big[\overset{u}{x^2}\, \overset{v}{\tfrac12 \mathrm{e}^{2x}}\Big]_0^z - \int_0^z \mathrm{d}x\, \overset{u'}{2x}\, \overset{v}{\tfrac12 \mathrm{e}^{2x}}$$

The integral on the right can be done by integrating by parts a second time, see (a):

$$I(z) \overset{\text{(a)}}{=} \boxed{\tfrac12 z^2\mathrm{e}^{2z} - \tfrac12 z\mathrm{e}^{2z} + \tfrac14\left[\mathrm{e}^{2z} - 1\right]}$$

$$I'(z) = \left[\tfrac12(2z+2z^2) - \tfrac12(1+2z) + \tfrac14 2\right]\mathrm{e}^{2z} \overset{\checkmark}{=} z^2\mathrm{e}^{2z} \qquad\qquad I(\tfrac12) \overset{\checkmark}{=} \tfrac{\mathrm{e}}{8} - \tfrac14$$

Since we integrated by parts twice, I' yields more involved cancellations than for (a).

(c) $$I(z) = \int_0^z \mathrm{d}x\, \overset{u}{(\ln x)} \cdot \overset{v'}{1} = \Big[\overset{u}{(\ln x)}\, \overset{v}{x}\Big]_0^z - \int_0^z \mathrm{d}x\, \overset{u'}{\tfrac1x}\, \overset{v}{x} = \boxed{(\ln z)z - z}$$

$$I'(z) = \tfrac1z z + \ln z - 1 \overset{\checkmark}{=} \ln z \qquad\qquad I(1) \overset{\checkmark}{=} -1$$

(d) $$I(z) = \int_0^z \mathrm{d}x\, \overset{u}{(\ln x)} \cdot \overset{v'}{\tfrac{1}{\sqrt{x}}} = \Big[\overset{u}{(\ln x)}\, \overset{v}{2\sqrt{x}}\Big]_0^z - \int_0^z \mathrm{d}x\, \overset{u'}{\tfrac1x}\, \overset{v}{2\sqrt{x}} = \boxed{(\ln z)2\sqrt{z} - 4\sqrt{z}}$$

To evaluate $[\ln(x)\sqrt{x}]_{x=0}$, we used the rule of L'Hôpital (see sheet 01, optional problems 3,4):

$$\Big[(\ln x)\sqrt{x}\Big]_{x=0} = \lim_{x\to 0}\frac{\ln x}{x^{-1/2}} = \lim_{x\to 0}\frac{\frac{\mathrm{d}}{\mathrm{d}x}\ln x}{\frac{\mathrm{d}}{\mathrm{d}x}x^{-1/2}} = \lim_{x\to 0}\frac{x^{-1}}{-\frac12 x^{-3/2}} = \lim_{x\to 0}\left[-2x^{1/2}\right] = \boxed{0}.$$

Thus the divergence of $\ln(x)$ for $x \to 0$ is so slow that $\sqrt{x}$ suppresses it.

$$I'(z) = 2\left[\tfrac1z\sqrt{z} + (\ln z)\tfrac12\tfrac{1}{\sqrt{z}}\right] - 4\tfrac12\tfrac{1}{\sqrt{z}} \overset{\checkmark}{=} (\ln z)\tfrac{1}{\sqrt{z}} \qquad\qquad I(1) \overset{\checkmark}{=} -4$$

(e) $$I(z) = \int_0^z \mathrm{d}x\, \overset{u}{\sin x}\, \overset{v'}{\sin x} = \Big[\overset{u}{\sin x}\, \overset{v}{(-\cos x)}\Big]_0^z - \int_0^z \mathrm{d}x\, \underbrace{\overset{u'}{\cos x}\, \overset{v}{(-\cos x)}}_{\sin^2 x - 1}$$

Reexpress the integral on the right in terms of $I(z)$,

$$I(z) = -\sin z\cos z - I(z) + \int_0^z \mathrm{d}x\ 1\ , \qquad \text{and solve for } I(z):$$

$$I(z) = \boxed{\tfrac{1}{2}(-\sin z\cos z + z)}$$

$$I'(z) = \tfrac{1}{2}(-\cos^2 z + \sin^2 z + 1) \overset{\checkmark}{=} \sin^2 z \qquad I(\pi) \overset{\checkmark}{=} \tfrac{\pi}{2}$$

(f)
$$I(z) = \int_0^z \mathrm{d}x\ \overset{u}{\sin^3 x}\ \overset{v'}{\sin x} = \Big[\overset{u}{\sin^3 x}\ (\overset{v}{-\cos x})\Big]_0^z - \int_0^z \mathrm{d}x\ \big(3\sin^2 x \overset{u'}{\cos x}\big)\ \underbrace{(\overset{v}{-\cos x})}_{\sin^2 x - 1}$$

Reexpress the integral on the right in terms of $I(z)$,

$$I(z) = -\sin^3 z\cos z - 3\Big[I(z) - \int_0^z \mathrm{d}x\ \sin^2 x\Big]\ , \qquad \text{solve for } I(z), \text{ and use (e):}$$

$$I(z) \overset{(e)}{=} \boxed{\tfrac{1}{4}\left[-\sin^3 z\cos z + \tfrac{3}{2}(-\sin z\cos z + z)\right]}$$

$$I'(z) = \tfrac{1}{4}\Big[-3\sin^2 z\underbrace{\cos^2 z}_{1-\sin^2 z} + \sin^4 z + \tfrac{3}{2}(-\cos^2 z + \sin^2 z + 1)\Big] \overset{\checkmark}{=} \sin^4 z \qquad I(\pi) \overset{\checkmark}{=} \tfrac{3\pi}{8}$$

Solution Example Problem 10: Integration by substitution [4]

(a)
$$I(z) = \int_0^z \mathrm{d}x\ x\cos(x^2+\pi) \qquad [y(x) = x^2,\ \mathrm{d}y = 2x\ \mathrm{d}x]$$

$$= \tfrac{1}{2}\int_{y(0)}^{y(z)} \mathrm{d}y\ \cos(y+\pi) = \tfrac{1}{2}\sin(y+\pi)\Big|_0^{z^2} = \boxed{\tfrac{1}{2}\sin(z^2+\pi)}$$

$$I'(z) = \tfrac{1}{2}\cos(z^2+\pi)\ \tfrac{\mathrm{d}}{\mathrm{d}z}z^2 \overset{\checkmark}{=} \cos(z^2+\pi)\ z \qquad I\big(\sqrt{\tfrac{\pi}{2}}\big) \overset{\checkmark}{=} -\tfrac{1}{2}$$

(b)
$$I(z) = \int_0^z \mathrm{d}x\ \sin^3 x\cos x \qquad [y(x) = \sin x,\ \mathrm{d}y = \cos x\ \mathrm{d}x]$$

$$= \int_{y(0)}^{y(z)} \mathrm{d}y\ y^3 = \tfrac{1}{4}y^4\Big|_0^{\sin z} = \boxed{\tfrac{1}{4}\sin^4 z}$$

$$I'(z) = \sin^3 z\ \tfrac{\mathrm{d}}{\mathrm{d}z}\sin z \overset{\checkmark}{=} \sin^3 z\cos z \qquad I(\tfrac{\pi}{4}) \overset{\checkmark}{=} \tfrac{1}{16}$$

(c)
$$I(z) = \int_0^z \mathrm{d}x \sin^3 x = \int_0^z \mathrm{d}x \sin x\big[1-\cos^2 x\big] \qquad [y(x) = \cos x,\ \mathrm{d}y = -\sin x\,\mathrm{d}x]$$

$$= -\int_{y(0)}^{y(z)} \mathrm{d}y\ (1-y^2) = -(y - \tfrac{1}{3}y^3)\Big|_1^{\cos z} = \boxed{-\cos z + \tfrac{1}{3}\cos^3 z + \tfrac{2}{3}}$$

$$I'(z) = \sin z + \cos^2 z(-\sin z) = \sin z(1-\cos^2 z) \overset{\checkmark}{=} \sin^3 z \qquad I(\tfrac{\pi}{3}) \overset{\checkmark}{=} \tfrac{5}{24}$$

(d)
$$I(z) = \int_0^z \mathrm{d}x \cosh^3 x = \int_0^z \mathrm{d}x \cosh x\big[1+\sinh^2 x\big] \quad [y(x) = \sinh x,\ \mathrm{d}y = \cosh x\mathrm{d}x]$$

$$= \int_{y(0)}^{y(z)} \mathrm{d}y\ (1+y^2) = (y + \tfrac{1}{3}y^3)\Big|_0^{\sinh z} = \boxed{\sinh z + \tfrac{1}{3}\sinh^3 z}$$

$$I'(z) = \cosh z + \sinh^2 z\cosh z = \cosh z(1+\sinh^2 z) \overset{\checkmark}{=} \cosh^3 z \qquad I(\ln 2) \overset{\checkmark}{=} \tfrac{57}{64}$$

(e) $$I(z) = \int_0^z \mathrm{d}x\ \sqrt{1+\ln(x+1)}\ \tfrac{1}{x+1} \qquad \big[y(x) = \ln(x+1),\ \mathrm{d}y = \tfrac{1}{1+x}\ \mathrm{d}x\big]$$
$$= \int_{y(0)}^{y(z)} \mathrm{d}y\ \sqrt{1+y} = \tfrac{2}{3}(1+y)^{3/2}\Big|_0^{\ln(z+1)} = \boxed{\tfrac{2}{3}\Big[\big(1+\ln(z+1)\big)^{3/2} - 1\Big]}$$
$$I'(z) = \big(1+\ln(z+1)\big)^{1/2}\tfrac{\mathrm{d}}{\mathrm{d}z}\ln(z+1) \ \overset{\checkmark}{=} \sqrt{1+\ln(z+1)}\ \tfrac{1}{z+1} \qquad I(\mathrm{e}^3-1) \overset{\checkmark}{=} \tfrac{14}{3}$$

(f) $$I(z) = \int_0^z \mathrm{d}x\ x^3\mathrm{e}^{-x^4} \qquad [y(x) = x^4,\ \mathrm{d}y = 4x^3\ \mathrm{d}x]$$
$$= \tfrac{1}{4}\int_{y(0)}^{y(z)} \mathrm{d}y\ \mathrm{e}^{-y} = -\tfrac{1}{4}\mathrm{e}^{-y}\Big|_0^{z^4} = \boxed{\tfrac{1}{4}\Big[1-\mathrm{e}^{-z^4}\Big]}$$
$$I'(z) = \tfrac{1}{4}\mathrm{e}^{-z^4}\tfrac{\mathrm{d}}{\mathrm{d}z}z^4 \ \overset{\checkmark}{=} \mathrm{e}^{-z^4}z^3 \qquad I(\sqrt[4]{\ln 2}) \overset{\checkmark}{=} \tfrac{1}{8}$$

[Total Points for Example Problems: 34]
