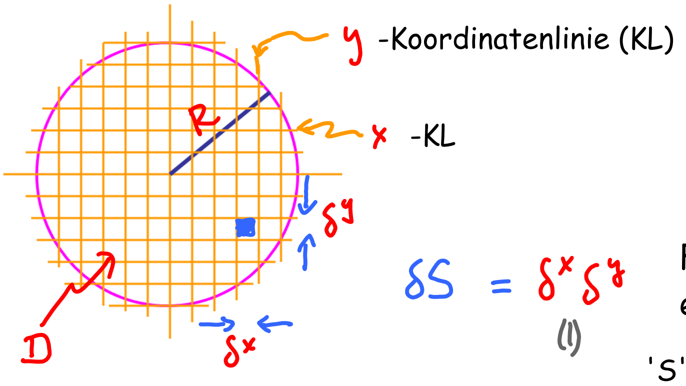
C4 Integration in krummlinigen Koordinaten

Falls ein System Symmetrien hat (z.B. invariant unter Rotationen um eine Symmetrie-Achse), lassen sich Integrale durch Nutzung krummliniger Koordinaten einfacher berechnen.

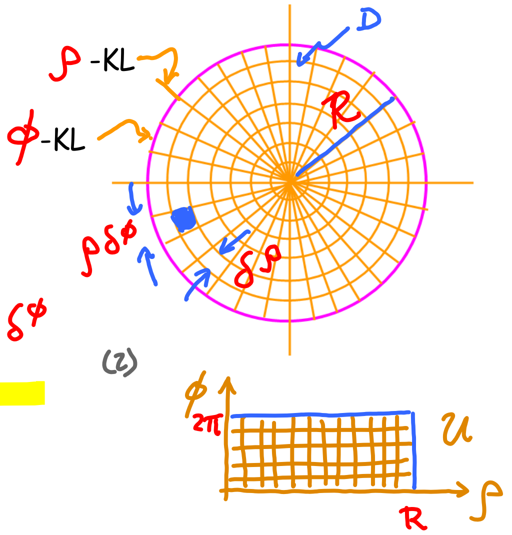
Beispiel: Kreisfläche



'D' für 'disk'

$\delta S = \delta^x \delta^y$ (1) Flächen-element $\delta S = \rho\, \delta^\rho \delta^\phi$ (2)

'S' für 'surface'



Nach geeigneter Koordinatentransformation wird aus der Scheibe

$D = \{(x,y)\in\mathbb{R}^2 : x^2+y^2 \le R^2\}$ ein 'Rechteck': $\mathcal{U} = \{(\rho,\phi)\in\mathbb{R}^2 : \rho \in (0,R);\ \phi \in (0,2\pi)\}$ (3)

Kreisfläche: $A = \int_S \overbrace{dx\,dy}^{(1)} \cdot 1 \overset{(2)}{=} \int_{\mathcal{U}} \overbrace{\rho\, d\phi\, d\rho}^{(2)} = \int_0^R d\rho \cdot \rho \int_0^{2\pi} d\phi = \frac{1}{2} R^2 \cdot 2\pi = \pi R^2$ (4)

Kartesisches Flächenelement — Polares Flächenelement

'A' für 'area' [eleganter als auf Seite C4e !]

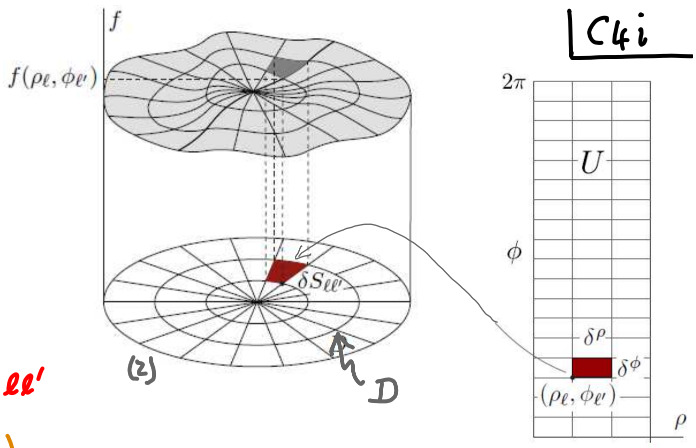
C4.2 2D-Integral in Polarkoordinaten

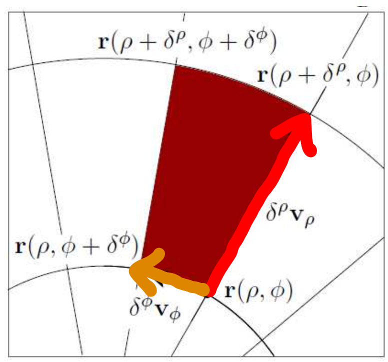
$$\int_D dS\, f(\vec r) \simeq \sum_{\ell\ell'} |\delta S_{\ell\ell'}|\, f(\rho_\ell, \phi_{\ell'}) \quad (1)$$

Flächenelement $\delta S_{\ell\ell'}$ wird aufgespannt durch:

$$\vec r(\rho_\ell + \delta^\rho, \phi_{\ell'}) - \vec r(\rho_\ell, \phi_{\ell'}) = \delta^\rho \partial_\rho \vec r(\rho_\ell, \phi_{\ell'}) \overset{(V2e.2)}{=} \delta^\rho (\vec v_\rho)_{\ell\ell'} \quad (2)$$

$$\vec r(\rho_\ell, \phi_{\ell'} + \delta^\phi) - \vec r(\rho_\ell, \phi_{\ell'}) = \delta^\phi \partial_\phi \vec r(\rho_\ell, \phi_{\ell'}) \overset{(V2e.4)}{=} \delta^\phi (\vec v_\phi)_{\ell\ell'} \quad (3)$$



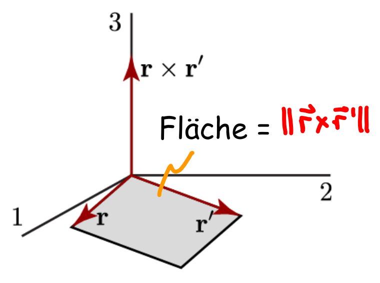


Geometrische Fläche von $\delta S_{\ell\ell'}$: $|\delta S_{\ell\ell'}| = \delta^\rho \delta^\phi \, \|(\vec v_\rho)_{\ell\ell'} \times (\vec v_\phi)_{\ell\ell'}\|$ (4)

$$\int_D dS\, f(\vec r) = \int_0^R d\rho \int_0^{2\pi} d\phi\, f(\rho, \phi)\, \underbrace{\|\vec v_\rho \times \vec v_\phi\|}_{|v_\rho v_\phi|\, \|\vec e_\rho \times \vec e_\phi\|} \quad (5)$$

(V2l.4): $v_\rho = 1$, $v_\phi = \rho$; $\underbrace{\|\vec e_\rho \times \vec e_\phi\|}_{=1}$ (V2l.5) (6)

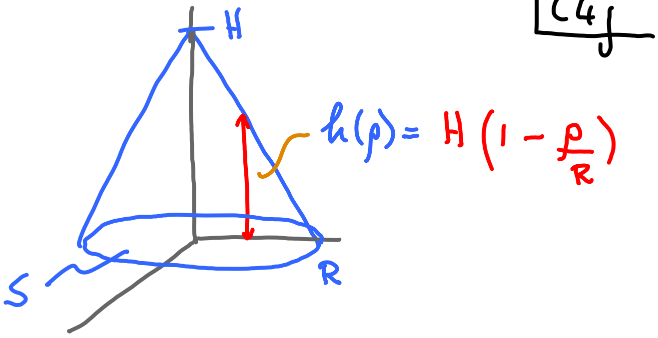
$$= \int_0^R \rho\, d\rho \int_0^{2\pi} d\phi\, f(\rho, \phi) \quad (7)$$



'Integrationsmaß' in Polarkoordinaten: '$dS = \rho\, d\rho\, d\phi$' (8)

C4j

Beispiel: Volumen eines Kegels



$$V = \int_D dS \; h(\rho) \qquad (1)$$

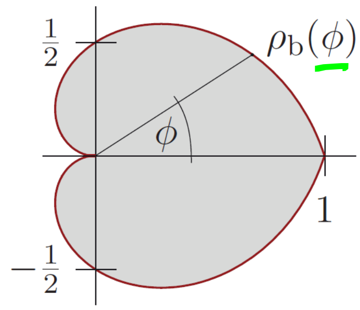
$$= \int_0^R d\rho \cdot \rho \int_0^{2\pi} d\phi \quad H\left(1 - \rho/R\right) \qquad (2)$$

$$= 2\pi H \int_0^R d\rho \left(\rho - \rho^2/R\right) = 2\pi H \left[\tfrac{1}{2}\rho^2 - \tfrac{1}{3}\rho^3/R\right]_0^R = 2\pi H \tfrac{1}{6}R^2 = \tfrac{1}{3}\pi H R^2 \qquad (3)$$

Beispiel: Herz-Fläche (geschachtelte Integrationsgrenzen)

Abstand vom Ursprung zum Rand des Herzens:

$$\rho_b(\phi) = \left(1 - \frac{|\phi|}{\pi}\right), \qquad \phi \in (-\pi, \pi) \qquad (4)$$

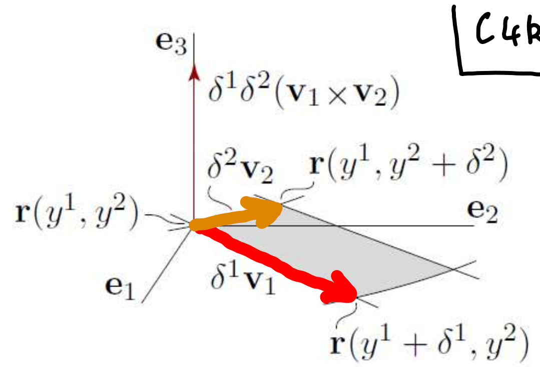


$$\text{Fläche} = \int_{-\pi}^{\pi} d\phi \int_0^{\rho_b(\phi)} d\rho \cdot \rho = \int_{-\pi}^{\pi} d\phi \; \tfrac{1}{2} \underbrace{\rho_b^2(\phi)}_{\left[1 - \frac{|\phi|}{\pi}\right]^2} \qquad (5)$$

$$= \tfrac{1}{3}\pi$$

C4k

Allgemeine Koordinatentransformation in 2D



$$\vec{r}: \; U \subset \mathbb{R}^2 \to M \subset \mathbb{R}^2$$

$$\vec{y} = \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} \mapsto \vec{r}(\vec{y}) = \begin{pmatrix} x^1(y^1, y^2) \\ x^2(y^1, y^2) \end{pmatrix} \qquad (1)$$

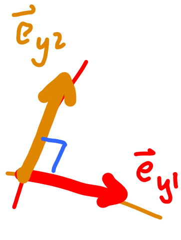
Flächenelement wird aufgespannt durch: $\delta^1 \vec{v}_1 = \delta^1 \partial_{y^1}\vec{r}, \quad \delta^2 \vec{v}_2 = \delta^2 \partial_{y^2}\vec{r}$ (2)

$$|\delta S| \overset{(L4b.1)}{=} \delta^1\delta^2 \|\vec{v}_1 \times \vec{v}_2\| = \delta^1\delta^2 \|\partial_{y^1}\vec{r} \times \partial_{y^2}\vec{r}\| \qquad (3)$$

2D-Integral: $\int_M dS \, f(\vec{r}) = \int_U dy^1 dy^2 \, \|\partial_{y^1}\vec{r} \times \partial_{y^2}\vec{r}\| \, f(\vec{r}(y^1, y^2))$ (4)

(V2f.4) $$\partial_{y^1}\vec{r} \times \partial_{y^2}\vec{r} = \underbrace{\|\vec{v}_{y^1}\|}_{\equiv v_{y^1} = \sqrt{g_{y^1y^1}}} \vec{e}_{y^1} \times \underbrace{\|\vec{v}_{y^2}\|}_{\equiv v_{y^2} = \sqrt{g_{y^2y^2}}} \vec{e}_{y^2} \qquad (5)$$

Für krummlinig-orthogonale Koordinaten gilt: $\|\vec{e}_{y^1} \times \vec{e}_{y^2}\| = 1$ (6)



Integrationsmaß: $dS = dy^1 dy^2 \; v_{y^1} v_{y^2}$ (7)

Bsp: Polarkoord.

$y^1 = \rho, \quad y^2 = \phi$

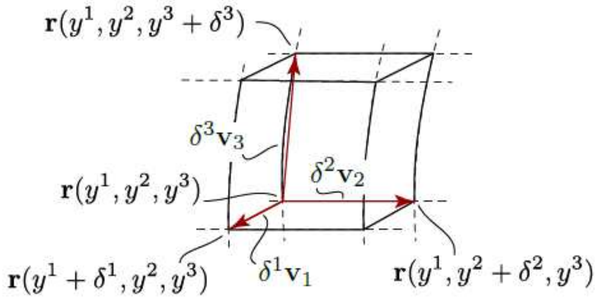
$v_{y^1} = 1, \quad v_{y^2} = \rho$

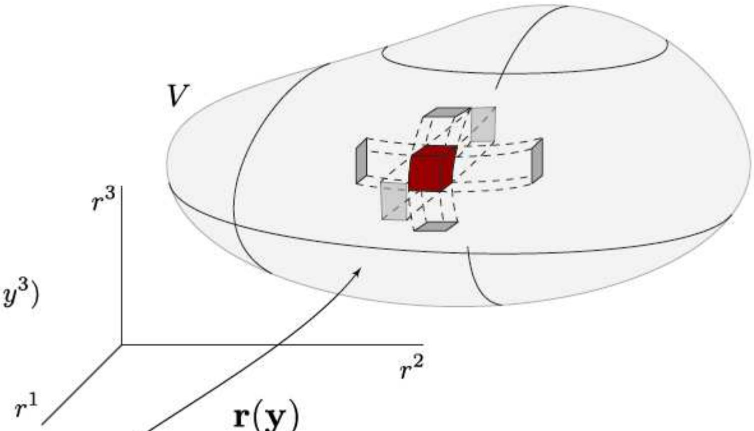
$\vec{e}_{y^1} = \vec{e}_\rho, \quad \vec{e}_{y^2} = \vec{e}_\phi$

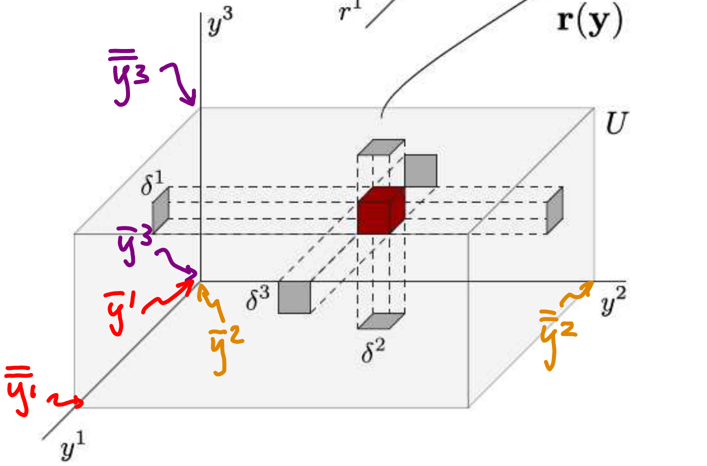
$\vec{e}_{y^1} \times \vec{e}_{y^2} = 1$

$dS \overset{(7)}{=} d\rho \, d\phi \cdot \rho$

C4.3 Volumenintegrale







$$\vec{r}: \mathcal{U} \subset \mathbb{R}^3 \longrightarrow V \subset \mathbb{R}^3$$

$$\vec{y} = \begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix} \longmapsto \vec{r}(\vec{y}) = \underset{\text{kartesisch}}{} \begin{pmatrix} x^1(y^1, y^2, y^3) \\ x^2(y^1, y^2, y^3) \\ x^3(y^1, y^2, y^3) \end{pmatrix}$$

$$\mathcal{U} = (\underline{y}^1, \bar{\bar{y}}^1) \times (\underline{y}^2, \bar{\bar{y}}^2) \times (\underline{y}^3, \bar{\bar{y}}^3)$$

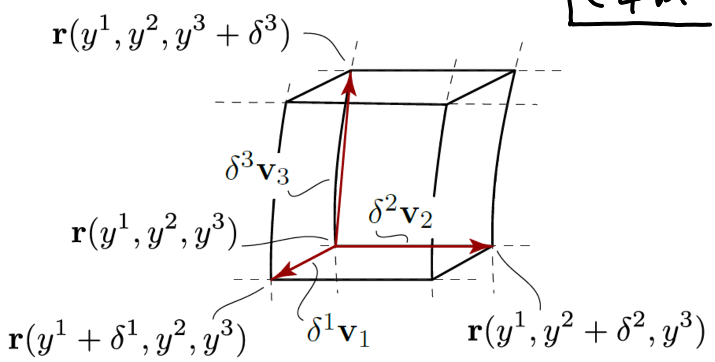
Bsp: Kugelkoordinaten: $y^1 = r,\ y^2 = \theta,\ y^3 = \phi$

Ball mit Radius R: $\mathcal{U} = (0, R) \times (0, \pi) \times (0, 2\pi)$

Allgemeine Koordinatentransformation in 3D

$$\vec{r}: \mathcal{U} \subset \mathbb{R}^3 \longrightarrow V \subset \mathbb{R}^3$$

$$\vec{y} \longmapsto \vec{r}(y^1, y^2, y^3) \qquad (1)$$



Volumenelement:

Volumen des Parallelelipeds = Spatprodukt (L4m.4):

$$\delta V = \left| (\vec{v}_1 \times \vec{v}_2) \cdot \vec{v}_3 \quad \delta^1 \, \delta^2 \, \delta^3 \right| \qquad (2)$$

3D-Integral:

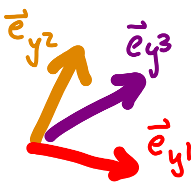
$$\int_V dV \, f(\vec{r}) = \int_{\underline{y}^1}^{\bar{\bar{y}}^1} dy^1 \int_{\underline{y}^2}^{\bar{\bar{y}}^2} dy^2 \int_{\underline{y}^3}^{\bar{\bar{y}}^3} dy^3 \, \left\| \left(\partial_{y^1} \vec{r} \times \partial_{y^2} \vec{r}\right) \cdot \partial_{y^3} \vec{r} \right\| \; f(\vec{r}(y^1, y^2, y^3)) \qquad (3)$$

(V2f.4):

$$\left(\partial_{y^1} \vec{r} \times \partial_{y^2} \vec{r}\right) \cdot \partial_{y^3} \vec{r} = \left(v_{y^1} \vec{e}_{y^1} \times v_{y^2} \vec{e}_{y^2}\right) \cdot \left(v_{y^3} \vec{e}_{y^3}\right) \qquad (4)$$

Für krummlinig-<u>orthogonale</u> Koordinaten gilt:

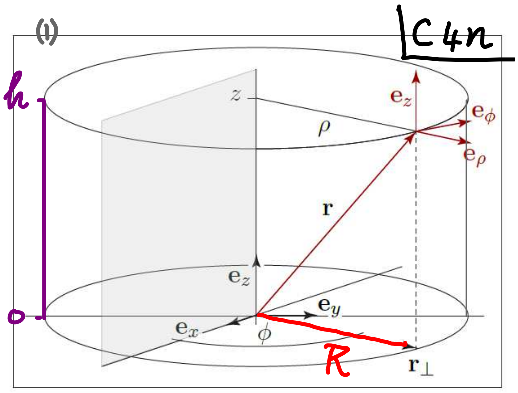
$$\left| \left(\vec{e}_{y^1} \times \vec{e}_{y^2}\right) \cdot \vec{e}_{y^3} \right| = 1 \qquad (5)$$



Integrationsmaß:

$$dV = dy^1 \, dy^2 \, dy^3 \, v_{y^1} \, v_{y^2} \, v_{y^3} \qquad (6)$$

Beispiel 1: Zylinderkoordinaten $\quad y^1 = \rho\,,\ y^2 = \phi\,,\ y^3 = z$ $\qquad$ (1)



(V2I.1): $\quad \vec{r} = \vec{e}_1\,\rho\cos\phi + \vec{e}_2\,\rho\sin\phi + \vec{e}_3\,z$ $\qquad$ (2)

(V2I.2): $\quad v_\rho = 1\,,\quad v_\phi = \rho\,,\quad v_z = 1$ $\qquad$ (3)

(C4m.6): $\quad dV = d\rho\, d\phi\, dz\, \rho$ $\qquad$ (4)

Volumen eines Zylinders: $\quad V = \int_0^R d\rho \cdot \rho \int_0^{2\pi} d\phi \int_0^h dz = \frac{1}{2}R^2 \cdot 2\pi \cdot h = \pi R^2 h$ ✓ $\qquad$ (5)

Trägheitsmoment eines homogenen Zylinders: $\quad I \equiv \int_V dV\, \rho(\vec{r})\, d_\perp^2(\vec{r})$ $\qquad$ (6)

Dichte = ρ_0 = Masse/Volumen = $\dfrac{M}{\pi R^2 h}$ $\qquad$ (7)

(nicht mit Radius zu verwechseln!)

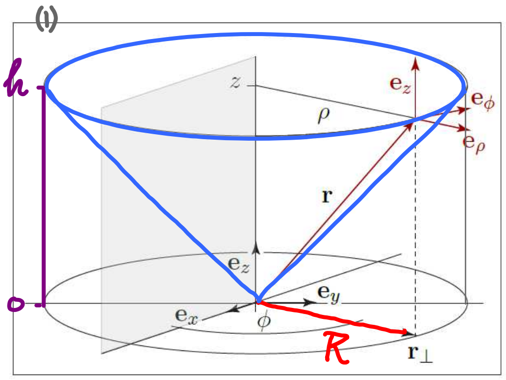
Abstand v. Symmetrieachse: $\quad d_\perp(\vec{r}) = \rho$ $\qquad$ (8)

$$I = \int_V dV\, \overbrace{\rho(\vec{r})}^{(6):\ \rho_0}\, \overbrace{d_\perp^2(\vec{r})}^{(7):\ \rho^2} \overset{(C4l.6)}{=} \int_0^R d\rho \cdot \rho \int_0^{2\pi} d\phi \int_0^h dz\ \rho_0 \cdot \rho^2 = \underbrace{\rho_0}_{(7)\ \frac{M}{\pi R^2 h}}\ \underbrace{\int_0^R d\rho\, \rho^3}_{\frac{1}{4}R^4}\ \underbrace{\int_0^{2\pi} d\phi \cdot 1}_{2\pi}\ \underbrace{\int_0^h dz \cdot 1}_{h} \qquad (9)$$

$$= \frac{1}{2} M R^2 \qquad (10)$$

Beispiel 2: Volumen eines Kegels

Zylinderkoordinaten: $\quad y^1 = \rho\,,\ y^2 = \phi\,,\ y^3 = z$ $\qquad$ (1)



(V2I.1): $\quad \vec{r} = \vec{e}_1\,\rho\cos\phi + \vec{e}_2\,\rho\sin\phi + \vec{e}_3\,z$ $\qquad$ (2)

(V2I.3): $\quad v_\rho = 1\,,\quad v_\phi = \rho\,,\quad v_z = 1$ $\qquad$ (3)

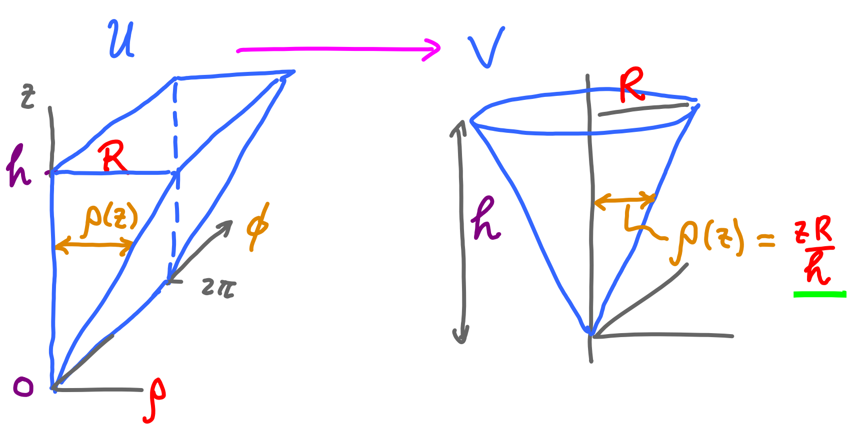
(C4m.6): $\quad dV = d\rho\, d\phi\, dz \cdot \rho$ $\qquad$ (4)

$$V = \int_0^h dz \int_0^{zR/h} d\rho \cdot \rho \int_0^{2\pi} d\phi$$

$$= \int_0^h dz \left[\frac{1}{2}\rho^2\right]_0^{zR/h} 2\pi$$

$$= \pi \left(\frac{R}{h}\right)^2 \underbrace{\int_0^h dz\ z^2}_{\frac{1}{3}h^3}$$

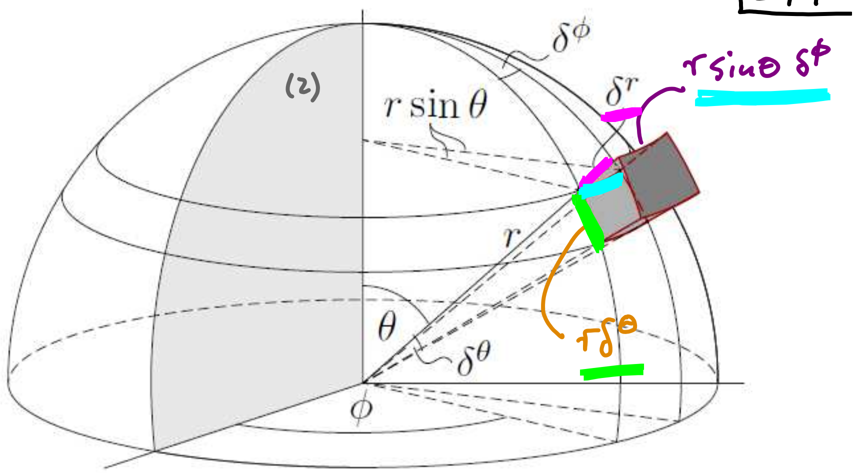
$$= \frac{\pi}{3} h R^2$$



Beispiel 3: Kugelkoordinaten

C4p

$$y^1 = r \,,\quad y^2 = \theta \,,\quad y^3 = \phi \qquad (1)$$



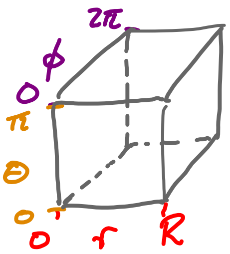
(V2n.1):
$$\vec{r} = \vec{e}_1 \, r \sin\theta \cos\phi + \vec{e}_2 \, r \sin\theta \sin\phi + \vec{e}_3 \, r \cos\theta \qquad (2)$$

(V2n.4:
$$v_r = \sqrt{g_{rr}} = 1 \,,\quad v_\theta = \sqrt{g_{\theta\theta}} = r \,,\quad v_\phi = \sqrt{g_{\phi\phi}} = r \sin\theta \qquad (3)$$

Integrationsmass:
$$dV \overset{(C4m.6)}{=} dr\, d\theta\, d\phi \cdot 1 \cdot r \cdot r\sin\theta = dr\, d\theta\, d\phi\, r^2 \sin\theta \qquad (4)$$

Volumen einer Kugel:

$$V = \int_0^R dr\, r^2 \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi = \frac{1}{3} R^3 \underbrace{\left[-\cos\theta\right]_0^\pi}_{2} 2\pi = \frac{4}{3}\pi R^3 \qquad (5)$$

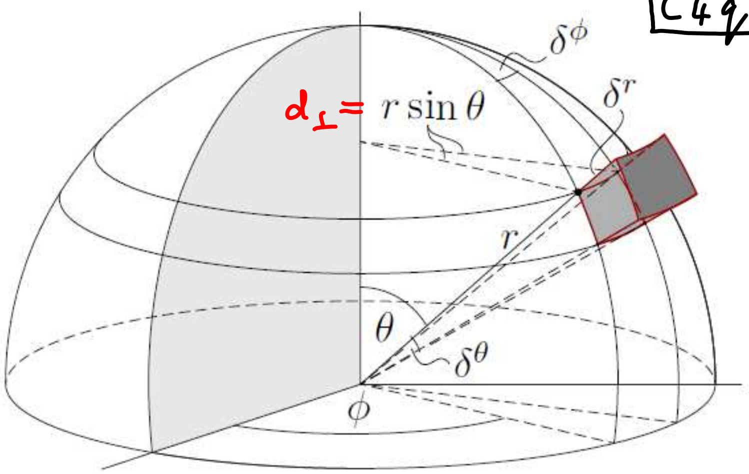


Trägheitsmoment einer homogenen Kugel:

C4q

Dichte = ρ_0 =Masse/Volumen = $M / \left(\frac{4\pi}{3} R^3\right)$ (6)

Abstand v. Symmetrieachse: $d_\perp(\vec{r}) = r \sin\theta$



$$I = \int_V dV \overbrace{\rho(\vec{r})}^{\rho_0} \overbrace{d_\perp^2(\vec{r})}^{(r\sin\theta)^2}$$

$$= \int_0^R dr\, r^2 \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \cdot \rho_0 \, r^2 (\sin\theta)^2$$

$$= \underbrace{\rho_0}_{\frac{M}{4\pi/3\, R^3}} \underbrace{\int_0^R dr\, r^4}_{\frac{1}{5} R^5} \underbrace{\int_0^\pi d\theta\, (\sin\theta)^3}_{\cos\theta = u:\ \int_{-1}^{1} du\,(1-u^2)} \underbrace{\int_0^{2\pi} d\phi\, 1}_{2\pi}$$

(6)

$$\int_{-1}^{1} du\,(1-u^2) = \left[u - \frac{1}{3}u^3\right]_{-1}^{1} = 2\left[1 - \frac{1}{3}\right] = \frac{4}{3}$$

$$= \frac{2}{5} M R^2$$

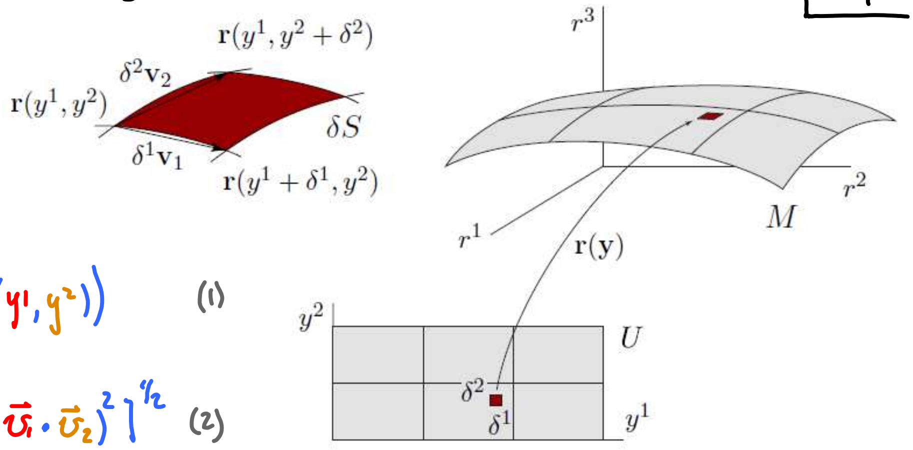
Standardsubstitution für Kugelkoordinaten:

$$\cos\theta = u$$
$$-\sin\theta = \frac{du}{d\theta}$$
$$\cos(0) = 1$$
$$\cos(\pi) = -1$$
$$\int_0^\pi d\theta \sin\theta = + \int_{-1}^{+1} du$$

C4.4 2D-Flächenintegrale in 3 Dimensionen (gekrümmte Flächen)

$$\vec r: \; U \subset \mathbb{R}^2 \to M \subset \mathbb{R}^3$$

$$\vec y = \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} \mapsto \vec r(\vec y) = \begin{pmatrix} x^1(\vec y) \\ x^2(\vec y) \\ x^3(\vec y) \end{pmatrix}$$



$$\int_M dS\, f(\vec r) \overset{(k.4)}{=} \int_U dy^1 dy^2 \, \|\partial_{y^1}\vec r \times \partial_{y^2}\vec r\| \, f(\vec r(y^1,y^2)) \qquad (1)$$

(analog zu Seite C4k)

$$\|\vec v_1 \times \vec v_2\| = \left[(\vec v_1)^2(\vec v_2)^2 - (\vec v_1 \cdot \vec v_2)^2\right]^{1/2} \qquad (2)$$

Lagrange-Identität (L4k.7)

Für krummlinig-orthogonale Koordinaten, mit $\|\vec v_1 \times \vec v_2\| = \|\vec v_1\| \, \|\vec v_2\| - 0 = \sqrt{g_{11}\, g_{22}}$ (3)

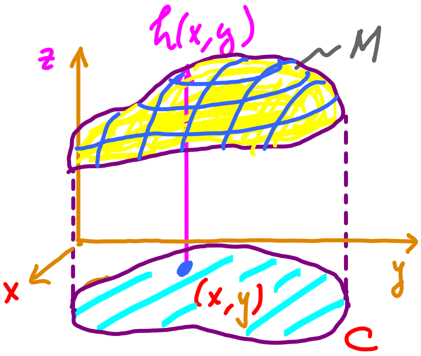
ist das Integrationsmaß: $dS = dy^1 dy^2 \, v_1 v_2 = dy^1 dy^2 \sqrt{g_{11}\, g_{22}}$ (4)

Beispiel 1: Oberfläche einer Kugel (Radius R)

$$dS \overset{(V2n.3)}{=} d\theta\, d\phi \cdot r \cdot r \sin\theta \qquad (5)$$

$$A = \int_{\text{Kugel}} d\theta\, d\phi \cdot \left[r^2 \sin\theta\right]_{r=R} = R^2 \int_0^{\pi} d\theta \, \sin\theta \int_0^{2\pi} d\phi \cdot 1 = R^2 \cdot 2 \cdot 2\pi = 4\pi R^2 \qquad (6)$$

Beispiel 2: Hügel $\quad y^1 = x, \quad y^2 = y$



$$\vec r(x,y) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ h(x,y) \end{pmatrix}, \qquad (1)$$

'Höhe des Hügels'

$$\vec v_1 = \partial_x \vec r = \begin{pmatrix} 1 \\ 0 \\ h_x \end{pmatrix}, \; h_x \equiv \frac{\partial h}{\partial x} \qquad \vec v_2 = \partial_y \vec r = \begin{pmatrix} 0 \\ 1 \\ h_y \end{pmatrix}, \; h_y \equiv \frac{\partial h}{\partial y} \qquad (2)$$

$$\|\partial_x \vec r \times \partial_y \vec r\| = \|\vec v_1 \times \vec v_2\| \overset{(r.2)}{=} \left[(\vec v_1)^2(\vec v_2)^2 - (\vec v_1 \cdot \vec v_2)^2\right]^{1/2} \qquad (3)$$

$$\overset{(1)}{=} \left[(1+h_x^2)(1+h_y^2) - (\underbrace{h_x h_y}_{\neq 0})^2\right]^{1/2} = \left[1 + h_x^2 + h_y^2\right]^{1/2} \qquad (4)$$

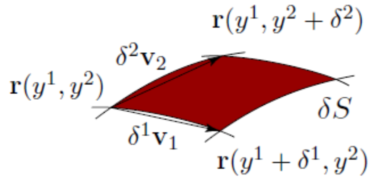
(also nicht-orthog. Koordinatensystem)

Wie groß ist 'Hügeloberfläche' über dem Bereich $(x,y) \in C$?

$$A_M = \int_M dS \overset{(r.1)}{=} \int_C dx\, dy \, \|\partial_x \vec r \times \partial_y \vec r\| \overset{(4)}{=} \int_C dx\, dy \left[1 + h_x^2 + h_y^2\right]^{1/2} \qquad (5)$$

Zusammenfassung: 2D-Flächenintegrale für Fläche in d = 2,3 Dimensionen

$$\vec r: U \subset \mathbb{R}^2 \to M \subset \mathbb{R}^n, \qquad \vec y = \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} \mapsto \vec r(\vec y) = \begin{pmatrix} x^1(\vec y) \\ \vdots \\ x^n(\vec y) \end{pmatrix} \qquad (1)$$

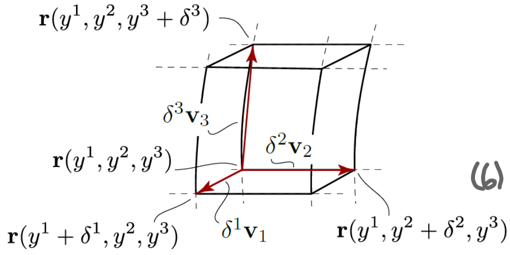


$$\int_M dS\, f(\vec r) = \int_U dy^1 \int dy^2\, \|\partial_{y^1}\vec r \times \partial_{y^2}\vec r\|\, f(\vec r(y^1, y^2)) \qquad (2)$$

für krummlinig-orthogonale Koord.: $$\|\partial_{y^1}\vec r \times \partial_{y^2}\vec r\| = v_{y^1} v_{y^2} = \sqrt{g_{11}\, g_{22}} \qquad (3)$$

Integrationsmaß: Polar: $dS = \rho\, d\rho\, d\phi$ (4) Kugel: $dS = d\theta\, d\phi\, r^2 \cdot \sin\theta$ (5)

Zusammenfassung: 3D Volumenintegrale



$$\vec r: U \subset \mathbb{R}^3 \to V \subset \mathbb{R}^3, \qquad \vec y \mapsto \vec r(\vec y) = \begin{pmatrix} x^1(\vec y) \\ x^2(\vec y) \\ x^3(\vec y) \end{pmatrix} \qquad (6)$$

$$\int_V dr^1 dr^2 dr^3\, f(\vec r) = \int_U dy^1 dy^2 dy^3\, \|(\partial_{y^1}\vec r \times \partial_{y^2}\vec r)\cdot \partial_{y^3}\vec r\|\, f(\vec r(y^1, y^2, y^3)) \qquad (7)$$

für krummlinig-orthogonale Koord.: $$\|(\partial_{y^1}\vec r \times \partial_{y^2}\vec r)\cdot \partial_{y^3}\vec r\| = v_{y^1} v_{y^2} v_{y^3} = \sqrt{g_{11}\, g_{22}\, g_{33}} \qquad (8)$$

Integrationsmaß: Zylinder: $dV = \rho\, d\rho\, d\phi\, dz$ (9) Kugel: $dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$ (10)