Ludwig-Maximilians-Universität München

SOLUTIONS TO

Quantum Field Theory (Quantum Electrodynamics)

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19 February 2021

Guidelines :

- The exam consists of 6 problems.
- The duration of the exam is 24 hours.
- Please write your name or matriculation number on every sheet that you hand in.
- Your answers should be comprehensible and readable.

GOOD LUCK !

Exercise 1	25 P
Exercise 2	20 P
Exercise 3	15 P
Exercise 4	15 P
Exercise 5	20 P
Exercise 6	5 P

Total	100 P

Problem 1 (25 points)

Take the following Lagrangian density in 4 spacetime dimensions (we use units $\hbar = c = 1$),

$$\mathcal{L} = i\bar{\psi}\gamma^\mu \partial_\mu \psi + \lambda((\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2) \ , \tag{1}$$

where ψ is a spinor and λ is a constant. As usual, $\bar{\psi} \equiv \psi^\dagger \gamma_0$.

a) [1P] What is the mass dimension of ψ?
Solution : From the kinetic term we read off $[\psi] = M^{3/2}$

b) [1P] What is the mass dimension of λ?
Solution : $[\lambda] = M^{-2}$

c) [4P] Find the equations of motion for the theory.
Solution :

$$\begin{aligned} \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi} &= \frac{\partial \mathcal{L}}{\partial \psi} \\ \Rightarrow i\partial_\mu \bar{\psi}\gamma^\mu &= 2\lambda \left[(\bar{\psi}\psi)\bar{\psi} - (\bar{\psi}\gamma_5\psi)\bar{\psi}\gamma_5 \right] (2P) \end{aligned}$$

$$\begin{aligned} \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \bar{\psi}} &= \frac{\partial \mathcal{L}}{\partial \bar{\psi}} \\ \Rightarrow i\gamma^\mu \partial_\mu \psi &= -2\lambda \left[(\bar{\psi}\psi)\psi - (\bar{\psi}\gamma_5\psi)\gamma_5\psi \right] (2P) \end{aligned}$$

d) [2P] Requiring that the Lagrangian is Lorentz invariant, can ψ be a Dirac spinor (in terms of degrees of freedom)?
Solution : Yes, because the action in (1) is Lorentz invariant, if ψ transforms as a Dirac spinor. In particular, also the bilinear including γ_5 is Lorentz invariant, because γ_5 commutes with the generators of the Lorentz group.

e) [2P] Requiring that the Lagrangian is Lorentz invariant, can ψ be a Majorana spinor?
Solution : Yes, since a Majorana spinor transforms in the same way as a Dirac spinor under Lorentz transformations.

f) [2P] Introduce left-handed ψ_L and right-handed ψ_R chiral spinors. How do they transform under $\psi \to \psi' = e^{i\alpha\gamma_5}\psi$, with α a nonzero real constant?
Solution :

$$\begin{aligned} \psi_{L,R} &\to e^{i\alpha\gamma_5}\psi_{L,R} \\ \Rightarrow \psi_L &\to e^{-i\alpha}\psi_L \ (1P) \\ \Rightarrow \psi_R &\to e^{i\alpha}\psi_R \ (1P) \end{aligned}$$

g) [6P] Write the Lagrangian in terms of the ψ_L and ψ_R spinors. Is it invariant under the above chiral transformation? If yes, find the corresponding Noether current. Check that it is conserved on the equations of motion.
Solution :
We will repeatedly use $\psi = \psi_L + \psi_R$ and $\bar{\psi} = \bar{\psi}_L + \bar{\psi}_R$ with $\bar{\psi}_{L,R} = \bar{\psi}_{L,R} P_{R,L}$, where $P_{R,L}$ are the right and left handed projectors.

$$\bar{\psi}\partial\!\!\!/\psi = \bar{\psi}_L\partial\!\!\!/\psi_L + \bar{\psi}_R\partial\!\!\!/\psi_R \ (1P)$$
$$\bar{\psi}\psi = \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L \ (1P)$$
$$\bar{\psi}\gamma_5\psi = \bar{\psi}_L\psi_R - \bar{\psi}_R\psi_L \ (1P)$$
$$\Rightarrow \mathcal{L} = i\bar{\psi}_L\partial\!\!\!/\psi_L + i\bar{\psi}_R\partial\!\!\!/\psi_R + 4\lambda(\bar{\psi}_L\psi_R)(\bar{\psi}_R\psi_L) \ (1P)$$

Check invariance. (0.5 P)
Solution : It is obvious that the above is indeed invariant under the chiral transformation.
The Noether current reads :

$$j^\mu = \frac{1}{\alpha}\frac{\partial\mathcal{L}}{\partial\partial_\mu\psi}\delta\psi = \bar{\psi}\gamma^\mu\gamma_5\psi \ (0.5P)$$

Check conservation. (1 P)
Solution :

$$\alpha\partial_\mu j^\mu = \partial_\mu\bar{\psi}\gamma^\mu\gamma_5\psi - \bar{\psi}\gamma_5\gamma^\mu\partial_\mu\psi \ . \tag{2}$$

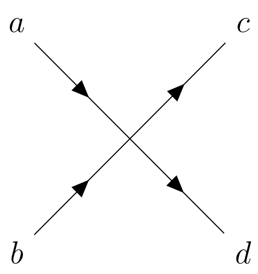
Plugging into the above the equations of motion for $\psi, \bar{\psi}$ we obtain

$$\alpha\partial_\mu j^\mu = -4i\lambda\left[(\bar{\psi}\psi)(\bar{\psi}\gamma_5\psi) - (\bar{\psi}\gamma_5\psi)(\bar{\psi}\psi)\right] = 0 \ . \tag{3}$$

h) [7P] Consider the $\psi\psi \ \rightarrow \ \psi\psi$ scattering process. Derive the spin-averaged amplitude squared at the leading order in λ.
Solution :
The interaction Hamiltonian is $H_{int} = -\lambda[(\bar{\psi}\psi)(\bar{\psi}\psi) - (\bar{\psi}\gamma_5\psi)(\bar{\psi}\gamma_5\psi)]$.
The following contractions will contribute to the process :

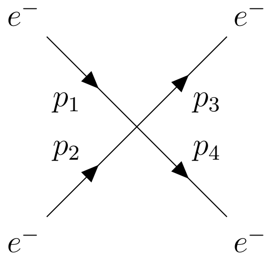
$$\langle e(p_3)e(p_4)| : \bar{\psi}\psi\bar{\psi}\psi : |e(p_1)e(p_2)\rangle - \langle e(p_3)e(p_4)| : \bar{\psi}\gamma_5\psi\bar{\psi}\gamma_5\psi : |e(p_1)e(p_2)\rangle$$

There is an additional set of contractions, if we interchange the contractions of ψ into initial states, which gives a minus-sign. We can also interchange the contractions of $\bar{\psi}$, but this is equivalent to interchanging the bilinears $\bar{\psi}\psi$, so it will contribute a factor 2! to the vertex. With this, we find the Feynman rule for the vertex



$$= 2i\lambda[\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc} - (\gamma_5)_{ac}(\gamma_5)_{bd} + (\gamma_5)_{ad}(\gamma_5)_{bc}] \ (2P)$$

where Latin letters denote spinor indices.
So, for the process



the Amplitude is

$$i\mathcal{M} = 2i\lambda[\bar{u}_3 u_1 \bar{u}_4 u_2 - \bar{u}_4 u_1 \bar{u}_3 u_2 - \bar{u}_3\gamma_5 u_1 \bar{u}_4 \gamma_5 u_2 + \bar{u}_4\gamma_5 u_1 \bar{u}_3\gamma_5 u_2] \ (1P)$$

where we use $u_i \equiv u(p_i, s_i)$.
We can simplify the calculation, if we split the amplitude as

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$$

with

$$\begin{aligned}\mathcal{M}_1 &\equiv 2\lambda(\bar{u}_3 u_1 \bar{u}_4 u_2 - \bar{u}_3\gamma_5 u_1 \bar{u}_4\gamma_5 u_2)\\ \mathcal{M}_2 &= -\mathcal{M}_1(3 \leftrightarrow 4)\end{aligned}$$

$$\Rightarrow |\mathcal{M}|^2 = |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 + (\mathcal{M}_1\mathcal{M}_2^* + \text{c.c.})$$

$$\begin{aligned}|\mathcal{M}_1|^2 =& 4\lambda^2\{\mathrm{Tr}(\mathrm{u}_4\bar{\mathrm{u}}_4\mathrm{u}_2\bar{\mathrm{u}}_2)\mathrm{Tr}(\mathrm{u}_1\bar{\mathrm{u}}_1\mathrm{u}_3\bar{\mathrm{u}}_3) + \mathrm{Tr}(\mathrm{u}_4\bar{\mathrm{u}}_4\gamma_5\mathrm{u}_2\bar{\mathrm{u}}_2\gamma_5)\mathrm{Tr}(\mathrm{u}_1\bar{\mathrm{u}}_1\gamma_5\mathrm{u}_3\bar{\mathrm{u}}_3\gamma_5)\\ &- [\mathrm{Tr}(\mathrm{u}_1\bar{\mathrm{u}}_1\gamma_5\mathrm{u}_3\bar{\mathrm{u}}_3)\mathrm{Tr}(\mathrm{u}_4\bar{\mathrm{u}}_4\mathrm{u}_2\bar{\mathrm{u}}_2\gamma_5) + \text{c.c.}]\} \ (1\mathrm{P})\end{aligned}$$

Now, using $\sum_{s_i} u_i\bar{u}_i = \not{p}_i$, we can derive

$$\begin{aligned}|\bar{\mathcal{M}}_1|^2 &= \frac{1}{4}\sum_{\{s\}}|\mathcal{M}_1|^2\\ &= \lambda^2[\mathrm{Tr}(\not{p}_4\not{p}_2)\mathrm{Tr}(\not{p}_1\not{p}_3) + \mathrm{Tr}(\not{p}_4\not{p}_2)\mathrm{Tr}(\not{p}_1\not{p}_3) + [\mathrm{Tr}(\not{p}_1\gamma_5\not{p}_3)\mathrm{Tr}(\not{p}_4\gamma_5\not{p}_2) + \text{c.c.}] \ (0.5\mathrm{P})\\ &= 32\lambda^2[(p_1p_3)(p_2p_4) \ (0.5P)\\ &= 8\lambda^2 t^2\end{aligned}$$

$$\begin{aligned}|\mathcal{M}_2|^2 &= |\mathcal{M}_1|^2(3 \leftrightarrow 4)\\ \Rightarrow |\bar{\mathcal{M}}_2|^2 &= 32\lambda^2(p_1p_4)(p_2p_3) \ (1P)\\ &= 8\lambda^2u^2\end{aligned}$$

A similar calculation leads to

$$\mathcal{M}_1\mathcal{M}_2^* = 0$$

$$\Rightarrow |\bar{\mathcal{M}}|^2 = 8\lambda^2(t^2 + u^2) \ (1P)$$

Problem 2 (20 points)

Consider the QED Lagrangian

$$\mathcal{L} = i\bar{\psi}\gamma^\mu D_\mu\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \ , \tag{4}$$

where

$$D_\mu \equiv \partial_\mu - ieA_\mu \ , \quad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\mu A_\nu \ . \tag{5}$$

a) [3P] What is the gauge redundancy of this Lagrangian ?
Solution :
The above Lagrangian is invariant under the following gauge transformation :

$$\psi \to \psi' = e^{-ie\alpha(x)}\psi \ , \quad A_\mu \to A'_\mu = A_\mu - \partial_\mu\alpha(x) \ . \tag{6}$$

b) [3P] How is this redundancy affected if we deform the theory by adding a mass term for the vector ?

$$\mathcal{L} = i\bar{\psi}\gamma^\mu D_\mu\psi - m\bar{\psi}\psi + \frac{1}{2}m_A^2\tilde{A}_\mu\tilde{A}^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \, . \tag{7}$$

Notice that a new notation is introduced in order to distinguish between the massive Proca field $\tilde{A}_\mu$ from the massless Maxwell field A_μ.
Solution : Once a mass term for the vector field is included, then this term is not invariant under the above gauge transformation - this can be explicitly checked.

c) [4P] How many degrees of freedom are propagated by A_μ and $\tilde{A}_\mu$? Explain.
Solution : The number of physical d.o.f. of the massless gauge field in 4 spacetime dimensions is equal to 2 (1P). The counting goes as follows :
We start initially with a field with 4 components. However, we have the Gauss law and the gauge redundancy (1 constraint + 1 invariance), giving in total 2 propagating d.o.f. (1P)
The number of physical d.o.f. of the massive gauge field in 4 spacetime dimensions is equal to 3 (1P). The counting goes as follows :
The equations of motion for the free massive vector read

$$\partial^\mu\tilde{F}_{\mu\nu} + m_A^2\tilde{A}_\nu = 0 \ . \tag{8}$$

Acting on the above with ∂^ν, we immediately obtain the following constraint

$$\partial^\mu\tilde{A}_\mu = 0 \ . \tag{9}$$

The equations of motion for the massive vector in the presence of a source term read

$$\partial^\mu\tilde{F}_{\mu\nu} + m_A^2\tilde{A}_\nu = j_\nu \ . \tag{10}$$

Acting on the above with ∂^ν, we immediately obtain the following constraint

$$\partial^\mu\tilde{A}_\mu = \frac{1}{m_A^2}\partial^\mu j_\mu \ . \tag{11}$$

In both cases, the physical d.o.f. for the massive vector are 3 (1P).

d) [4P] Can the massive theory be written in manifestly gauge redundant (gauge invariant) form ? If yes, write it.

Solution : Yes, the theory can be written in a manifestly gauge invariant form (1P). To achieve that, we start from the Proca theory and write $\tilde{A}_\mu = A_\mu + \partial_\mu \phi$, where ϕ is a scalar field that under a gauge transformation behaves as

$$\phi \to \phi' = \phi + \alpha \ . \tag{12}$$

We find (3P)

$$\mathcal{L} \supset -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \partial_\mu \chi \partial^\mu \chi + m_A A^\mu \partial_\mu \chi + \frac{m_A^2}{2} A_\mu A^\mu + A^\mu j_\mu + \frac{1}{m_A} j_\mu \partial^\mu \chi \ , \tag{13}$$

with $\chi \equiv m_A \phi$.

e) [2P] Write down the Feynman rule for the vertex of the theory in equation (4).

Solution : This is exactly the same as the vertex in the conventional QED.

f) [4P] What are the possible polarization states of the massive Proca field ?

Solution : The massive Proca field of course has 3 polarizations (2P). These correspond to the possible projections of the spin on the z-axis, i.e. $S_z = -1, 0, 1$. To find the polarization vectors, we first take the particle at rest and look at the eigenvectors of the generator of 3-dimensional spatial rotations around the z-axis :

$$T_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ . \tag{14}$$

The corresponding eigenvectors read

$$\vec{\epsilon}_1 = \frac{1}{\sqrt{2}}(1, i, 0), \ \vec{\epsilon}_2 = \frac{1}{\sqrt{2}}(1, -i, 0), \vec{\epsilon}_3 = (0, 0, 1), \tag{15}$$

therefore, the polarization four-vectors are (2P)

$$\epsilon_i^\mu = (0, -\vec{\epsilon}_i) \ . \tag{16}$$

Problem 3 (15 points)

Consider a theory with two real scalar fields, ϕ and χ, with the following Lagrangian

$$\mathcal{L} = \frac{3}{2}\partial_\mu\phi\partial^\mu\phi + \frac{3}{2}\partial_\mu\chi\partial^\mu\chi + \partial_\mu\phi\partial^\mu\chi - m^2(\phi^2+\chi^2)\ . \tag{17}$$

a) [5P] Find the Lagrangian for canonically normalized fields.
Solution : To find the Lagrangian for the canonically normalized fields, we have to get rid of the kinetic mixing between ϕ and χ. This is achieved by rotating the fields as(2P for finding redefinition+1P for correct 1/2 coefficient+2P for Lagrangian and masses)

$$\phi = \frac{1}{\sqrt{2}}\left(\frac{\psi_1}{2}+\frac{\psi_1}{\sqrt{2}}\right)\ ,\quad \chi = \frac{1}{\sqrt{2}}\left(\frac{\psi_1}{2}-\frac{\psi_1}{\sqrt{2}}\right)\ . \tag{18}$$

Substituting the above into the Lagrangian, we obtain

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2\ , \tag{19}$$

with

$$\begin{aligned}\mathcal{L}_1 &= \frac{1}{2}\partial_\mu\psi_1\partial^\mu\psi_1 - \frac{m_1^2}{2}\psi_1^2\ ,\\ \mathcal{L}_2 &= \frac{1}{2}\partial_\mu\psi_2\partial^\mu\psi_2 - \frac{m_2^2}{2}\psi_2\ ,\end{aligned} \tag{20}$$

and $m_1^2 = m^2/2, m_2^2 = m^2$.

b) [4P] Quantize this theory and write down the canonical commutation relations.
Solution : When the fields are canonically normalized, the Lagrangian captures the dynamics of two decoupled free massive fields. Therefore, we can quantize the two pieces independently like we did in PS 4, ex. 2 :
Start from the expansion of the fields and momenta in terms of creation and annihilation operators(2P)

$$\begin{aligned}\psi_i(t,\vec{x}') &= \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}_i}}\left(\hat{a}_i(\vec{p})e^{-i(\omega_{\vec{p}_i}t-\vec{p}\cdot\vec{x})} + \hat{a}_i^+(-\vec{p})e^{i(\omega_{\vec{p}_i}t+\vec{p}\cdot\vec{x})}\right)\ ,\\ \pi_i(t,\vec{x}') &= -\frac{i}{2}\int\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\left(\hat{a}_i(\vec{p})e^{-i(\omega_{\vec{p}_i}t-\vec{p}\cdot\vec{x})} - \hat{a}_i^+(-\vec{p})e^{i(\omega_{\vec{p}_i}t+\vec{p}\cdot\vec{x})}\right)\ ,\end{aligned} \tag{21}$$

Plug the above into the canonical commutation relations

$$[\hat{\psi}_i(t,\vec{x}),\hat{\psi}_j(t,\vec{x}')] = 0,\ [\hat{\pi}_i(t,\vec{x}),\hat{\pi}_j(t,\vec{x}')] = 0,\ [\hat{\psi}_i(t,\vec{x}),\hat{\pi}_j(t,\vec{x}')] = i\delta_{ij}\delta^{(3)}(\vec{x}-\vec{x}'), \tag{22}$$

to obtain(2P)

$$[\hat{a}_i(\vec{p}),\hat{a}_j(\vec{p}')] = [\hat{a}_i^+(\vec{p}),\hat{a}_j^+(\vec{p}')] = 0,\ [\hat{a}_i(\vec{p}),\hat{a}_j^+(\vec{p}')] = 2\omega_{\vec{p}_i}(2\pi)^3\delta_{ij}\delta^{(3)}(\vec{p}-\vec{p}')\ . \tag{23}$$

c) [6P] Express the Hamiltonian in terms of creation and annihilation operators.
Solution : We have derived in details the Hamiltonian for free fields in PS 4, ex. 2. For the theory under consideration here, (6P)

$$H = \int\frac{\mathrm{d}^3\vec{p}}{2(2\pi)^3}\sum_{i=1,2}\hat{a}_i^+(\vec{p})\hat{a}_i(\vec{p})\ . \tag{24}$$

Problem 4 (15 points)

For two spinors, ψ and χ, consider the following quantities

$$\begin{aligned}
&\text{A) } \bar{\psi}_L \chi_R \\
&\text{B) } \bar{\psi}_L \gamma_\mu \chi_R \\
&\text{C) } \bar{\psi}_L \gamma_\mu \partial^\mu \chi_R \\
&\text{D) } \bar{\psi}_L \gamma_\mu \partial^\mu \chi_L \\
&\text{E) } \bar{\psi}_L \chi_L \\
&\text{F) } (\bar{\psi}_L \gamma_\mu \chi_L)(\bar{\psi}_L \gamma^\mu \chi_L) \\
&\text{G) } (\bar{\psi}_L \gamma_\mu \chi_L)(\bar{\psi}_R \gamma^\mu \chi_R)
\end{aligned} \tag{25}$$

where L, R denote the chiralities (corresponding to the ± 1 eigenvalues of the γ_5 matrix) and for any spinor X, $\bar{X} \equiv X^\dagger \gamma_0$.

a) Which of the above quantities are identically zero ?

Solution : Some useful relations are :

$$\bar{\psi}_{L,R} = \bar{\psi} P_{R,L} \ , \quad \gamma_\mu P_{L,R} = P_{R,L} \gamma_\mu \ , \quad P_{L,R} \gamma_5 = \mp P_{L,R} \ . \tag{26}$$

$$\begin{aligned}
&A)\ \bar{\psi}_L \chi_R = \bar{\psi} P_R^2 \chi = \bar{\psi} \chi_R \neq 0 \\
&B)\ \bar{\psi}_L \gamma_\mu \chi_R = \bar{\psi} P_R \gamma_\mu P_R \chi = \bar{\psi} P_R P_L \gamma_\mu \chi = 0 (2P) \\
&C)\ \bar{\psi}_L \gamma_\mu \partial^\mu \chi_R = \bar{\psi} P_R \gamma_\mu \partial^\mu P_R \chi = 0 (2P) \\
&D)\ \bar{\psi}_L \gamma_\mu \partial^\mu \chi_L = \bar{\psi} P_R \gamma_\mu \partial^\mu P_L \chi = \bar{\psi} \gamma_\mu \partial^\mu \chi_L \neq 0 \\
&E)\ \bar{\psi}_L \chi_L = 0 (2P) \\
&F)\ (\bar{\psi}_L \gamma_\mu \chi_L)(\bar{\psi}_L \gamma^\mu \chi_L) \neq 0 \\
&G)\ (\bar{\psi}_L \gamma_\mu \chi_L)(\bar{\psi}_R \gamma^\mu \chi_R) \neq 0
\end{aligned} \tag{27}$$

b) Which of them can be non-zero and Lorentz-scalars ?

Solution : It is clear from the above that the bilinears A), D), F) and G) are the only nonzero Lorentz scalars(8P, 2P for each).

c) Which of them can be non-zero and Lorentz-vectors ?

Solution : there is no nonzero Lorentz vector.(1P)

Justify your answer in each case.

Problem 5 (20 points)

Consider the following Lagrangian in $d = 4$ spacetime dimensions (we use units $\hbar = c = 1$)

$$\mathcal{L} = \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} M^2 \Phi^2 - \frac{1}{2} m^2 \phi^2 - \kappa \Phi \phi^2 \ , \tag{28}$$

which involves two scalar fields Φ and ϕ with masses M and m, respectively, and κ is a constant.

a) [1P] What is the mass dimension of κ?

$$
\begin{aligned}
[\mathcal{L}] &= M^4 \\
[\Phi] &= [\phi] = M \\
[\mathcal{L}] &\overset{!}{=} [\kappa][\Phi][\phi]^2 \\
\Leftrightarrow \boxed{[\kappa]} &\boxed{= M} \quad (1\text{P})
\end{aligned}
$$

b) [4P] What are the conditions on the masses of the particles such that a particle of type Φ can decay into two particles of type ϕ?

For the process $\Phi \to \phi\phi$ to be kinematically allowed :

$$\boxed{M \geq 2m} \quad (4\text{P})$$

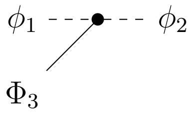
c) [4P] Write down the Feynman rules for this theory.
The Feynman rules can be written in either real space, or momentum space.
Feynman rules

1. (1P) Propagators :

$$x \overset{\Phi}{\text{———}} y \;= D^F_\Phi(x,y) = \lim_{\varepsilon\to 0^+}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i}{p^2 - M^2 + i\varepsilon}\mathrm{e}^{ip\cdot(x-y)}$$

$$x \overset{\phi}{\text{- - -}} y \;= D^F_\phi(x,y) = \lim_{\varepsilon\to 0^+}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i}{p^2 - m^2 + i\varepsilon}\mathrm{e}^{ip\cdot(x-y)}$$

2. (2P) Vertex :

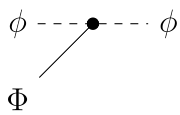


$$H_{\text{int}} = \int \mathrm{d}^3x\, \mathcal{H}_{\text{int}} = \kappa \int \mathrm{d}^3x\, \Phi\phi\phi$$

$$\langle 0|\,\mathrm{T}\left\{\Phi(x_3)\left[1 - i\int \mathrm{d}t\, \mathcal{H}_{\text{int}}(t) + \cdots\right]\phi(x_1)\phi(x_2)\right\}|0\rangle \simeq$$

$$\simeq -i\kappa \int \mathrm{d}^4z\; \langle 0| \left(\Phi_3\Phi_z\phi_z\phi_z\phi_1\phi_2\right) + \left(\Phi_3\Phi_z\phi_z\phi_z\phi_1\phi_2\right)|0\rangle = -2i\kappa$$

Therefore, for the vertex we obtain



$$= (-2i\kappa)\int \mathrm{d}^4z$$

3. (1P for 3.-4.) External points :

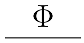
$$x \overset{\Phi}{\text{———}} = 1$$

$$x \overset{\phi}{\text{- - - -}} = 1$$

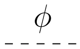
4. Divide by the symmetry factor

Momentum-space Feynman rules

1. (1P) Propagators :

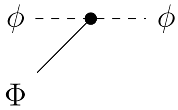


$$= \frac{i}{p^2 - M^2 + i\varepsilon}$$



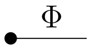
$$= \frac{i}{p^2 - m^2 + i\varepsilon}$$

2. (2P) Vertex :

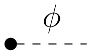


$$= -2i\kappa$$

3. (1P for 3.-6.) External points :



$$= \mathrm{e}^{-ip\cdot x}$$

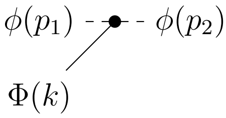


$$= \mathrm{e}^{-ip\cdot x}$$

4. Impose four-momentum conservation at each vertex
5. Integrate over each undetermined momentum : $\int \frac{\mathrm{d}^4 p}{(2\pi)^4}$
6. Divide by the symmetry factor

d) [6P] Consider the decay $\Phi \to \phi\phi$. Draw the Feynman diagram(s) contributing to this process at tree-level (lowest order). Use it to derive the expression for the amplitude squared.

Feynman diagram (1P) :



Field decomposition (1P) :

$$\Phi(x) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} \left[\hat{a}(\vec{p})\mathrm{e}^{-ip\cdot x} + \hat{a}^\dagger(\vec{p})\mathrm{e}^{ip\cdot x}\right]$$

$$\phi(x) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} \left[\hat{b}(\vec{p})\mathrm{e}^{-ip\cdot x} + \hat{b}^\dagger(\vec{p})\mathrm{e}^{ip\cdot x}\right]$$

States (1P) :

$$|i\rangle = |\vec{k}\rangle = \hat{a}^\dagger(\vec{k})\,|0\rangle$$

$$|f\rangle = |\vec{p}_1, \vec{p}_2\rangle = \hat{b}^\dagger(\vec{p}_1)\hat{b}^\dagger(\vec{p}_2)\,|0\rangle$$

Equation for transition amplitude (1P) :

$$i(2\pi)^4\delta^{(4)}(p_1 + p_2 - k)\mathcal{M} = -i\kappa\,\langle f|\int \mathrm{d}^4x\,\Phi\phi\phi\,|i\rangle =$$

Calculation (2P) :

$$= -2i\kappa(2\pi)^4\delta^{(4)}(p_1 + p_2 - k)$$

$$\Leftrightarrow \mathcal{M} = -2\kappa$$

$$\Leftrightarrow \boxed{|\mathcal{M}|^2 = 4\kappa^2}$$

e) [5P] Compute the lifetime of the particle Φ to lowest order in κ.
Lifetime τ is related to the decay rate Γ by

$$\tau = \frac{1}{\Gamma}$$

The decay rate is given by (1P) :

$$\Gamma = \frac{1}{2\omega_{\vec{k}}}\left(\frac{1}{2}\right)\int \frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^3 2\omega_{\vec{p}_1}}\int \frac{\mathrm{d}^3\vec{p}_2}{(2\pi)^3 2\omega_{\vec{p}_2}}(2\pi)^4\delta^{(4)}(p_1 + p_2 - k)|\mathcal{M}|^2$$

where the factor of $\left(\frac{1}{2}\right)$ comes from the symmetry due to identical ϕ's.

Center of mass frame, four-momentum conservation :

$$\delta^{(4)}(p_1 + p_2 - k) = \delta(\omega_{\vec{p}_1} + \omega_{\vec{p}_2} - M)\delta^{(3)}(\vec{p}_1 + \vec{p}_2)$$

$$\omega_{\vec{k}} = M$$

$$\omega^2_{\vec{p}_i} = m^2 + \vec{p}_i^{\,2};\ \ i = 1, 2$$

$$p \equiv |\vec{p}| : \quad \frac{\mathrm{d}\omega}{\mathrm{d}p} = \frac{1}{2\omega}2p \quad \Leftrightarrow \quad p\mathrm{d}p = \omega\mathrm{d}\omega$$

The decay rate then becomes (1P) :

$$\Gamma = \frac{1}{4M}4\kappa^2\int \frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^3 2\omega_{\vec{p}_1}}\frac{(2\pi)}{2\omega_{\vec{p}_1}}\underbrace{\delta(2\omega_{\vec{p}_1} - M)}_{=\frac{1}{2}\delta(\omega_{\vec{p}_1} - \frac{M}{2})}$$

Go to spherical coordinates for integraion over $\vec{p}_1$ (1P) :

$$\Gamma = \frac{\kappa^2}{M}\int \frac{\overbrace{\mathrm{d}p_1 \cdot p_1}^{=\mathrm{d}\omega_{\vec{p}_1}\cdot\omega_{\vec{p}_1}}\ p_1\ \overbrace{\mathrm{d}\Omega}^{=4\pi}}{(2\pi)^2 8\omega^2_{\vec{p}_1}}\delta\left(\omega_{\vec{p}_1} - \frac{M}{2}\right)$$

Using $p_1 = (\omega_{\vec{p}_1}^2 - m^2)^{1/2}$, we obtain (1P) :

$$\Gamma = \frac{\kappa^2}{8M} \frac{4\pi}{(2\pi)^2} \int \frac{\mathrm{d}\omega_{\vec{p}_1}}{\omega_{\vec{p}_1}} (\omega_{\vec{p}_1}^2 - m^2)^{1/2} \delta\left(\omega_{\vec{p}_1} - \frac{M}{2}\right) =$$

$$= \frac{\kappa^2}{8\pi M} \frac{\left(\frac{M^2}{4} - m^2\right)^{1/2}}{M/2} = \frac{\kappa^2}{8\pi M} \sqrt{1 - \frac{4m^2}{M^2}}$$

Finally, the lifetime is given by (1P) :

$$\tau = \frac{1}{\Gamma} = \frac{8\pi M}{\kappa^2} \frac{1}{\sqrt{1 - \frac{4m^2}{M^2}}}$$

Problem 6 (5 points)

Under what circumstances can a massless particle decay into two other particles ? Explain.
(5P) for a justification that makes sense ! :-)