Ludwig-Maximilians-Universität München

Quantum Field Theory (Quantum Electrodynamics)

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19 February 2021

Guidelines :

- The exam consists of 6 problems.
- The duration of the exam is 24 hours.
- Please write your name or matriculation number on every sheet that you hand in.
- Your answers should be comprehensible and readable.

GOOD LUCK!

Exercise 1	25 P
Exercise 2	20 P
Exercise 3	15 P
Exercise 4	15 P
Exercise 5	20 P
Exercise 6	5 P

Total	100 P

Problem 1 (25 points)

Take the following Lagrangian density in 4 spacetime dimensions (we use units $\hbar = c = 1$),

$$\mathcal{L} = i\bar{\psi}\gamma^\mu \partial_\mu \psi + \lambda((\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2) \ , \tag{1}$$

where ψ is a spinor and λ is a constant. As usual, $\bar{\psi} \equiv \psi^\dagger \gamma_0$.

a) What is the mass dimension of ψ ?

b) What is the mass dimension of λ ?

c) Find the equations of motion for the theory.

d) Requiring that the action is Lorentz invariant, can ψ be a Dirac spinor (in terms of degrees of freedom) ?

e) Requiring that the action is Lorentz invariant, can ψ be a Majorana spinor ?

f) Introduce left-handed ψ_L and right-handed ψ_R chiral spinors. How do they transform under $\psi \to \psi' = e^{i\alpha\gamma_5}\psi$, with α a nonzero real constant ?

g) Write the Lagrangian density in terms of the ψ_L and ψ_R spinors. Is it invariant under the above chiral transformation ? If yes, find the corresponding Noether current. Check that it is conserved on the equations of motion.

h) Consider the $\psi\psi \ \to \ \psi\psi$ scattering process. Derive the spin-averaged amplitude squared at the leading order in λ.

Problem 2 (20 points)

Consider the QED Lagrangian density

$$\mathcal{L} = i\bar{\psi}\gamma^\mu D_\mu \psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \ , \tag{2}$$

where

$$D_\mu \equiv \partial_\mu - ieA_\mu \ , \quad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\mu A_\nu \ . \tag{3}$$

a) What is the gauge redundancy of this Lagrangian ?

b) How is this redundancy affected if we deform the theory by adding a mass term for the vector ?

$$\mathcal{L} = i\bar{\psi}\gamma^\mu D_\mu \psi - m\bar{\psi}\psi + \frac{1}{2}m_A^2 \tilde{A}_\mu \tilde{A}^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \, . \tag{4}$$

Notice that a new notation is introduced in order to distinguish between the massive Proca field $\tilde{A}_\mu$ from the massless Maxwell field A_μ.

c) How many degrees of freedom are propagated by A_μ and $\tilde{A}_\mu$? Explain.

d) Can the massive theory be written in manifestly gauge redundant (gauge invariant) form ? If yes, write it.

e) Write down the Feynman rule for the vertex of the theory in equation (4).

f) What are the possible polarization states of the massive Proca field ?

Problem 3 (15 points)

Consider a theory with two real scalar fields, ϕ and χ, with the following Lagrangian density

$$\mathcal{L} = \frac{3}{2}\partial_\mu\phi\partial^\mu\phi + \frac{3}{2}\partial_\mu\chi\partial^\mu\chi + \partial_\mu\phi\partial^\mu\chi - m^2(\phi^2 + \chi^2) \ . \tag{5}$$

a) Find the Lagrangian density for canonically normalized fields.

b) Quantize this theory and write down the canonical commutation relations.

c) Express the Hamiltonian in terms of creation and annihilation operators.

Problem 4 (15 points)

For two spinors, ψ and χ, consider the following quantities

$$\begin{aligned}
&\text{A)}\ \bar{\psi}_L\chi_R \\
&\text{B)}\ \bar{\psi}_L\gamma_\mu\chi_R \\
&\text{C)}\ \bar{\psi}_L\gamma_\mu\partial^\mu\chi_R \\
&\text{D)}\ \bar{\psi}_L\gamma_\mu\partial^\mu\chi_L \\
&\text{E)}\ \bar{\psi}_L\chi_L \\
&\text{F)}\ (\bar{\psi}_L\gamma_\mu\chi_L)(\bar{\psi}_L\gamma^\mu\chi_L) \\
&\text{G)}\ (\bar{\psi}_L\gamma_\mu\chi_L)(\bar{\psi}_R\gamma^\mu\chi_R)
\end{aligned} \tag{6}$$

where L, R denote the chiralities (corresponding to the ± 1 eigenvalues of the γ_5 matrix) and for any spinor X, $\bar{X} \equiv X^\dagger\gamma_0$.

a) Which of the above quantities are identically zero ?

b) Which of them can be non-zero and Lorentz-scalars ?

c) Which of them can be non-zero and Lorentz-vectors ?

Justify your answer in each case.

Problem 5 (20 points)

Consider the following Lagrangian density in $d = 4$ spacetime dimensions (we use units $\hbar = c = 1$)

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi + \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}M^2\Phi^2 - \frac{1}{2}m^2\phi^2 - \kappa\Phi\phi^2 \ , \tag{7}$$

which involves two scalar fields Φ and ϕ with masses M and m, respectively, and κ is a constant.

a) What is the mass dimension of κ?

b) What are the conditions on the masses of the particles such that a particle of type Φ can decay into two particles of type ϕ?

c) Write down the Feynman rules for this theory.

d) Consider the decay $\Phi \rightarrow \phi\phi$. Draw the Feynman diagram(s) contributing to this process at tree-level (lowest order). Use it to derive the expression for the amplitude squared.

e) Compute the lifetime of the particle Φ to lowest order in κ.

Problem 6 (5 points)

Under what circumstances can a massless particle decay into two other massless particles? Explain.