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Gauge Theories Without Anomalies*

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Theories with spontaneously broken gauge symmetry but without triangle anomalies can yield renormalizable models of weak and electromagnetic interactions. We identify and discuss the class of anomaly-free gauge theories.

Weinberg[1] and others[2,3] have suggested the possibility of constructing a renormalizable and realistic theory of weak and electromagnetic interactions. They propose a class of models in which the vector boson mediating weak interactions gets its mass by way of the Higgs phenomenon.[4] These models are not naively renormalizable by power counting since they involve massive vector bosons coupled to nonconserved currents, but they can be obtained by formal transformations on the fields from apparently renormalizable gauge theories. The original gauge theories involve fields which do not correspond to physical particles, so they are not obviously unitary. However, as 't Hooft has emphasized, if a theory which is unitary but not obviously renormalizable is equivalent to a theory which is renormalizable but not obviously unitary, then they must both be unitary and renormalizable.[5] Hence the theories involving massive intermediate vector bosons will be renormalizable if they are equivalent to renormalizable gauge theories.[6]

The formal transformations connecting the original gauge theories with the intermediate vector boson theories can be justified in a path-integral formulation of field theory if the gauge theories can be renormalized in a gauge-invariant way. Gross and Jackiw[7] have shown that if the gauge invariance is spoiled by the presence of Adler-Bell-Jackiw anomalies,[8] these transformations are not valid and the resulting intermediate vector boson theory is not renormalizable. Thus, if a model of weak and electromagnetic interactions based on the Higgs phenomenon is to be renormalizable, it must be anomaly-free. The same comments apply if the model is extended to include strong interactions mediated by a massive vector gluon.

In this paper, we discuss the construction of anomaly-free gauge theories in general. The question of how the ideas we present fit into realistic theories will be treated elsewhere.[9]

Consider the gauge-invariant interaction of a multiplet of gauge fields W^a_μ with a multiplet of spinor fermion fields ψ,

$$gF^a_{\mu\nu}W^b_\mu W^c_\nu f_{abc} + gW^a_\mu \bar{\psi}\gamma_\mu \Gamma_a \psi ,$$

where the f_{abc} are the structure constants of the underlying Lie algebra, the Γ_a are Hermitian matrices which generate a representation of the algebra, and $F^a_{\mu\nu}$ stands for the covariant curl of W^a_μ. The Lie algebra is most generally the direct sum of simple Lie algebras and of completely Abelian factors. The f_{abc} may be chosen to be completely antisymmetric, and must satisfy the Jacobi identity,

$$f_{abc}f_{dcg} + f_{bdc}f_{acg} + f_{dac}f_{bcg} = 0 .$$

The matrices Γ_a may involve γ_5, and must satisfy the commutation relations of the Lie algebra,

$$[\Gamma_a, \Gamma_b] = if_{abc}\Gamma_c .$$

The triangle graph with three W vertices has an anomaly given[7] by a positive multiple of

$$A_{abc} = \mathrm{Tr}(\gamma_5\{\Gamma_a, \Gamma_b\}\Gamma_c) .$$

A theory is completely free of anomalies if and only if all the triangle-graph anomalies are absent.[7] Thus, we shall study the circumstances under which A_{abc} is identically zero.

It is convenient to write Γ_a in terms of left- and right-handed parts,

$$\Gamma_a = \tfrac{1}{2}(1+\gamma_5)T^+_a + \tfrac{1}{2}(1-\gamma_5)T^-_a ,$$

where the $T^\pm_a$ are numerical Hermitian matrices with the same commutation relations as the Γ_a,

$$[T^\pm_a, T^\pm_b] = if_{abc}T^\pm_c .$$

Then $A_{abc} = 2(A^+_{abc} - A^-_{abc})$ where

$$A^\pm_{abc} = \mathrm{Tr}(\{T^\pm_a, T^\pm_b\}T^\pm_c) .$$

We distinguish three varieties of anomaly-free models.

Case 1. $A^+ = A^- \neq 0$. The right-handed and left-handed anomalies cancel. This cancellation may take place naturally in the sense that

$$T^-_a = UT^+_a U^\dagger ,$$

where U is a fixed unitary matrix. Evidently $A^+ = A^-$ and the anomaly is zero. In this case the interaction of the gauge fields with the fermions may be rewritten in terms of vector currents alone,

$$\bar{\psi}\gamma_\mu[\tfrac{1}{2}(1+\gamma_5)T_a^+ + \tfrac{1}{2}(1-\gamma_5)T_a^-]\psi = \bar{\psi}'\gamma_\mu T_a^+\psi',$$

where

$$\psi' = \tfrac{1}{2}(1+\gamma_5)\psi + \tfrac{1}{2}(1-\gamma_5)U\psi .$$

We call such models "vector-like." The redefinition of ψ inserts γ_5 into the mass terms of the Lagrangian, where they play no role in producing an anomaly.

Case 1 can also be realized accidentally: A^+ can equal A^- without T^+ and T^- being equivalent. For example, imagine a one-parameter gauge theory with

$$T^+ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad T^- = \begin{bmatrix} (2)^{1/3} & 0 \\ 0 & 0 \end{bmatrix}.$$

In this case, $A^+ = A^- = 2$. Similar accidental cancellations may take place in non-Abelian models.

Case 2. $A^+ = A^- = 0$. We call a representation "safe" if its generators satisfy

$$\mathrm{Tr}(\{T_a, T_b\}T_c) = 0 .$$

If both T^+ and T^- generate safe representations, there is no anomaly. We can easily find a large class of safe representations. A representation is said to be real if it is equivalent to its complex conjugate; otherwise it is said to be complex.[10] If T_a are the generators of a real representation, then

$$T_a = -U T_a^* U^\dagger$$

for some fixed unitary matrix U. *All real representations are safe,* since

$$\mathrm{Tr}(\{T_a, T_b\}T_c) = -\mathrm{Tr}(\{T_a^*, T_b^*\}T_c^*)$$
$$= -\mathrm{Tr}(\{T_a, T_b\}T_c) .$$

A safe representation need not be real. As an example, consider a one-parameter gauge theory with

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -(2)^{1/3} \end{bmatrix}.$$

We will find less trivial examples of this phenomenon under Case 3.

Case 3. Safe algebras. This case is already included in Case 2, but deserves further discussion. There are some Lie algebras that can never yield anomalies: all their representations are safe. For example, SU(2) has only real representations: every representation is equivalent to its conjugate. Thus, SU(2) is safe.

Let us first consider the simple Lie algebras. We will show in the Appendix that the following simple Lie algebras are safe:

SU(2),
SO(N) for $N \geq 5$, $N \neq 6$,
Sp($2N$) for $N \geq 3$,
G(2), F(4), E(7), and E(8).

The algebras SU(N) for $N \geq 3$ are not safe. With the possible exception of $E(6)$, these are the only unsafe simple Lie algebras.

The most general safe gauge algebra is a direct sum of simple, safe algebras, with no completely Abelian components. Models based on these safe algebras cannot have anomalies no matter how the fermions transform under the gauge group.

Unsafe algebras, like SU(3), can also be used to construct anomaly-free theories, but only if the left- and right-handed anomalies cancel (Case 1) or if the representations involved are safe [Case 2; for SU(3), the $\underline{8}$ or $\underline{3}+\underline{\bar{3}}$ are examples of safe representations].

So far, we have discussed only anomalies trilinear in the W's. If there is a vector gluon, there will also be triangle anomalies involving one gluon and two W's. Let us suppose that the gluon coupling is

$$f G_\mu \bar{\psi}\gamma_\mu P\psi ,$$

where P is a numerical matrix not involving γ_5 which commutes with the gauge algebra; that is,

$$[P, T_a^+] = [P, T_a^-] = 0 .$$

The anomaly involving one gluon is given by

$$A_{ab} = \mathrm{Tr}(P\{T_a^+, T_b^+\} - P\{T_a^-, T_b^-\}) .$$

We will consider in detail the important special case in which P is a projection operator and the fermions separate into "leptons" with $P=0$ and "hadrons" with $P=1$. Then we can write

$$T_a^\pm = L_a^\pm + H_a^\pm ,$$

where

$$L_a^\pm = (1-P)T_a^\pm \text{ and } H_a^\pm = PT_a^\pm .$$

Then the fermions transform under the gauge group as a direct sum of lepton and hadron representations generated by $L^\pm$ and $H^\pm$, respectively. In this case, the anomaly is simply

$$A_{ab} = \mathrm{Tr}(\{H_a^+, H_b^+\} - \{H_a^-, H_b^-\}) .$$

Since $\mathrm{Tr}\{H_a^\pm, H_b^\pm\}$ is positive semidefinite, the only way to avoid an anomaly is to obtain a cancellation of the left-handed and right-handed terms. Although accidental cancellations are again possible, the most natural anomaly-free models are those in which H^+ and H^- are equivalent. As far as the hadrons are concerned, the model must be vector-like.

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APPENDIX

Mehta and Srivastava[11] have classified the irreducible representations of the simple Lie groups according to their reality properties. They find that the following simple Lie groups have only real representations:

SU(2)
SO($2N+1$) for $N \geq 2$
SO($4N$) for $N \geq 2$
Sp($2N$) for $N \geq 3$
G(2), F(4), E(7), and E(8).

Hence all these groups are safe. The groups SO($4N+2$) for $N \geq 1$ have complex representations; but except for SO(6), they are all safe. To show this, we give a simple proof that SO(N) for $N \geq 7$ is safe.

We label the generators of SO(N) by $T_{ij} = -T_{ji}$, $i = 1, 2, \ldots, N$. Then the commutation relations are

$$[T_{ij}, T_{kl}] = -i(\delta_{jk}T_{il} - \delta_{ik}T_{jl} + \delta_{jl}T_{ki} - \delta_{il}T_{kj}).$$

T_{ij} transforms like an antisymmetric tensor in the i and j indices. Then $\mathrm{Tr}(\{T_{ij}, T_{kl}\}T_{mn})$ is an invariant tensor; so it must be a linear combination of products of Kronecker δ's. Furthermore, it must be antisymmetric under the exchanges $i \leftrightarrow j$, $k \leftrightarrow l$, or $m \leftrightarrow n$ and symmetric under the exchange of pairs $ij \leftrightarrow kl$, $kl \leftrightarrow mn$, or $ij \leftrightarrow mn$. But such an object must be zero. The most general form consistent with the antisymmetry in $i \leftrightarrow j$, $k \leftrightarrow l$, and $m \leftrightarrow n$ is

$$a(\delta_{jk}\delta_{lm}\delta_{ni} - \delta_{ik}\delta_{lm}\delta_{nj} - \delta_{jl}\delta_{km}\delta_{ni} + \delta_{il}\delta_{km}\delta_{nj} - \delta_{jk}\delta_{ln}\delta_{mi} + \delta_{ik}\delta_{ln}\delta_{mj} + \delta_{jl}\delta_{kn}\delta_{mi} - \delta_{il}\delta_{kn}\delta_{mj}).$$

But this is antisymmetric in $ij \leftrightarrow kl$, so a must vanish. Therefore

$$\mathrm{Tr}(\{T_{ij}, T_{kl}\}T_{mn}) = 0$$

and the group is safe. This proof fails for $N = 6$, since then the trace may involve the completely antisymmetric tensor of rank 6.

To see that SU(N) for $N \geq 3$ are not safe, we consider the representations generated by the Hermitian, traceless $N \times N$ matrices. One of the generators can be taken to be the diagonal "hypercharge,"

$$T_Y = \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & -2 & & & & \\ & & & 0 & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & \cdot \end{bmatrix}.$$

Since $\mathrm{Tr}(T_Y{}^3) \neq 0$, *none* of these groups are safe. Since SU(4) is isomorphic to SO(6), we understand in retrospect why SO(6) is unsafe.

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[9]See Ref. 2 and H. Georgi and S. L. Glashow (unpublished).
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