Neutrino Mass and Grand Unification

Lecture XXVIII

8/2/2022

LMU
Winter 2022

SO(10) GUT

My notes:

1) Neutrino Mass and Grand Unification

→ Appendix D = Spinors

2) Families from Spinors

Wilczek, Zee 1982

- SM : $G_{SM} = SU(3)_c \times SU(2)_L \times U(1)_Y$

$$\boxed{SU(5) \supseteq G_{SM}}$$

• LR :

$$G_{LR} = SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$$

$$SU(4)_c \supseteq SU(3)_c \times U(1)_{B-L}$$

$$\begin{pmatrix} u & u & u & \nu \\ d & d & d & e \end{pmatrix}$$

Pati, Salam '73 - '74

$SU(4)_c$

$$SU(4)_c \xrightarrow{M_{PS}} SU(3)_c \times U(1)_{B-L}$$

(a) $SU(2) \times SU(2)$ "=" $SO(4)$

$r = 2$ $\qquad\qquad$ $r = 2$

$n = 3 + 3 = 6$ $\qquad\qquad$ $n = \frac{4 \cdot 3}{2} = 6$

(b) $SU(4)$ "$=$" $SO(6)$

$r = 3$ $\quad\quad$ $r = 3$

$n = 4^2 - 1 = 15$ $\quad\quad$ $n = \frac{6 \cdot 5}{2} = 15$

$$\underbrace{\underbrace{SU(2) \times SU(2)}_{\shortparallel} \times \underbrace{SU(4)}_{\shortparallel}}$$

$$= \underbrace{SO(4) \times SO(6)}$$

minimal GUT $= SO(10) \supseteq SO(4) \times SO(6)$

vector of $SO(10) = 10 = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_{10} \end{pmatrix}$

$$\left.\begin{array}{l}\left.\begin{array}{c}\phi_1\\ \vdots\\ \phi_6\end{array}\right\} SO(6)\\ \\ \left.\begin{array}{c}\phi_7\\ \phi_8\\ \phi_9\\ \phi_{10}\end{array}\right\} SO(4)\end{array}\right\} SO(10)$$

$$\boxed{SO(2N)}$$

$$\phi \to O\,\phi \qquad \therefore \quad OO^T = O^T O = 1$$

$$\det O = 1$$

$$O = e^{i\,\theta_{ij} L_{ij}}$$

$$(L_{ij})_{kl} = -i\,(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$$

$$L^{\dagger} = L, \quad L^{T} = -L, \quad L^{*} = -L$$

$$L_{ij} = -L_{ji} \qquad n = \frac{2N(2N-1)}{2}$$

$$[L_{ij}, L_{kl}] = i\left(\delta_{ik} L_{jl} - \delta_{il} L_{jk}\right)$$

$$L_{12} = \begin{pmatrix} \begin{matrix} 0 & -i \\ i & 0 \end{matrix} & 0 \\ 0 & 0 \end{pmatrix}$$

$$L_{34} = \begin{pmatrix} 0 & & 0 \\ & \begin{matrix} 0 & -i \\ i & 0 \end{matrix} & \\ 0 & & 0 \end{pmatrix}$$

$$[L_{12}, L_{34}] = [L_{12}, L_{56}] = \cdots = 0$$

$$C = \{L_{12}, L_{34}, \ldots, L_{2N-1,2N}\}$$

$$N$$

$$\boxed{r(SO(2N)) = N}$$

$$V \times V = S + A$$

$$\phi \times \phi \qquad \text{(symmetric)} \qquad \text{(anti symmetric)}$$

$$\tfrac{1}{2}\, 2N(2N+1) \qquad \tfrac{1}{2}\, 2N \cdot (2N-1)$$

$$SO(10)\,! \quad \underbrace{A : 10\cdot 9/_2 = 45}_{\text{adjoint}}$$

$$S = \frac{10\cdot 11}{2} = 55$$

$$S: \quad \phi_i \phi_j \to O_{ik} \phi_k O_{jl} \phi_l$$

$$S_{ij} \to O_{ik} S_{kl} O^T_{lj}$$

$$\boxed{\begin{aligned} S &\longrightarrow O S O^T \\ A &\longrightarrow O A O^T \end{aligned}} \Rightarrow \boxed{\mathrm{Tr}\, A = 0}$$

$$\mathrm{Tr}\, S \longrightarrow \mathrm{Tr}\, O S O^T = \mathrm{Tr}\, O^T O S = \mathrm{Tr}\, S$$

S : by subtracting $\mathrm{Tr}\, S$

$$n_S : \quad \frac{2N(2N+1)}{2} - 1$$

SO(10) : S → 54 elements

$$\boxed{\text{Spinors of } SO(2N)}$$

↓

$$\boxed{Spin(2N) \text{ "=" } SO(2N)}$$

$$\begin{cases} \Gamma_i & i = 1, \dots, 2N \\ \{\Gamma_i, \Gamma_j\} = 2\delta_{ij} \end{cases}$$

$$\Sigma_{ij} = \frac{1}{4i}\left[\Gamma_i, \Gamma_j\right]$$

$$\left[\Sigma_{ij}, \Sigma_{ke}\right] = i\left(\delta_{ik}\Sigma_{je} - - \right)$$

SO(2N) commutation

$\psi \to S\psi$ spinor

$S = e^{i\theta_{ij}\Sigma_{ij}}$, $\Sigma_{ij}^{+} = \Sigma_{ij}$

$\Gamma_i = \Gamma_i^{+}$

$$\left\{ \begin{array}{l} \Gamma_{FIVE} = (-i)^N \Gamma_1 \Gamma_2 \cdots \Gamma_{2N} \quad (\pm) \\ \therefore \quad \Gamma_{FIVE}^2 = 1 \end{array} \right\}$$

$$\{\Gamma_{FIVE}, \Gamma_i\} = 0$$

$$\Rightarrow [\Gamma_{FIVE}, \Gamma_i \Gamma_j] = \Gamma_{FIVE}\Gamma_i\Gamma_j - \Gamma_i\Gamma_j\Gamma_{FIVE}$$

$$= \Gamma_{FIVE}\,\Gamma_i\,\Gamma_j + \Gamma_i\,\Gamma_{FIVE}\,\Gamma_j$$

$$- \Gamma_i\,\Gamma_{FIVE}\,\Gamma_j - \Gamma_i\,\Gamma_j\,\Gamma_{FIVE}$$

$$= \{\Gamma_{FIVE}, \Gamma_i\}\,\Gamma_j - \Gamma_i\,\{\Gamma_j, \Gamma_{FIVE}\} = 0$$

$$\Downarrow$$

$$\boxed{[\Gamma_{FIVE}, \Sigma_{ij}] = 0}$$

$$\Rightarrow \quad \Gamma_\pm = \frac{1 \pm \Gamma_{FIVE}}{2}$$

$$\boxed{\begin{aligned} \psi_+ &\equiv \Gamma_+\,\psi \\ \psi_- &\equiv \Gamma_-\,\psi \end{aligned}} \qquad \left.\begin{matrix} (L) \\ (R) \end{matrix}\right\} \underline{\text{Lorentz}}$$

- $\Gamma_i' = S^+ \, \Gamma_i \, S \qquad (S^+ S = 1)$

$$\{\Gamma_i', \Gamma_j'\} = S^+ \{\Gamma_i, \Gamma_j\} S =$$

$$= 2\delta_{ij} \, S^+ S = 2\delta_{ij}$$

$$\Rightarrow \Gamma_i' = R_{ij} \, \Gamma_j$$

$$\{\Gamma_i', \Gamma_j'\} = R_{ik} R_{jl} \{\Gamma_k, \Gamma_l\}$$

$$\| $$

$$2\delta_{ij} \qquad = 2 R_{ik} R_{jk}$$

$$\Rightarrow R_{ik} R_{jk} = \delta_{ik}$$

$$(R R^T)_{ik} = \delta_{ij}$$

$$\boxed{R = O}$$

$$\boxed{\Gamma_i' = O_{ij} R_j}$$

old and new Gammas

Lorentz $\psi^T C \psi = \text{inv.}$

$\updownarrow$ analogy

$$\psi^T B \psi = \text{invariant}$$

$\downarrow$

$$\psi^T S^T B S \psi = \psi^T B S^+ S \psi = \text{inv.}$$

$$\boxed{S^T B = B S^+}$$

$$\boxed{\Sigma^T B = B \Sigma^+}$$

$\Downarrow$

$$\psi^T S^T B S \psi = \psi^T (1 + i\theta \Sigma^T) B (1 + i\theta \Sigma) \psi + \cdots$$

$$= \psi^T B \psi + i\theta\, \psi^T \underbrace{(\Sigma^T B + B\Sigma)}_{=0} \psi + \cdots$$

$$\boxed{\Sigma_{ij}^T B + B \Sigma_{ij} = 0}$$

$$\Gamma_{FIVE} = (-i)^N \Gamma_1 \cdots \Gamma_{2N}$$

$$= 2\Sigma_{12}\, 2\Sigma_{34} \cdots 2\Sigma_{2N-1,2N}$$

$$2\Sigma_{ij} = (-1)\,\Gamma_i\,\Gamma_j \quad (i \neq j)$$

$$\boxed{(2\Sigma_{ij})^2 = +1}$$

$\Sigma_{12}, \Sigma_{34}, \ldots$: ± 1 eigenvalues

denoted as ε_i

- $\Gamma_{FIVE} = 1 \Leftrightarrow \psi = \psi_+$

$$= 2\Sigma_{12} \ldots$$

$$= \varepsilon_1\,\varepsilon_2 \cdots \varepsilon_N$$

$$\boxed{\psi_+ = |\varepsilon_1\,\varepsilon_2 \cdots \varepsilon_N\rangle \quad \therefore\ \varepsilon_1 \cdots \varepsilon_N = +1}$$

$$\psi_- = |\varepsilon_1 \, \varepsilon_2 \, \cdots \, \varepsilon_N\rangle$$

$$: \; \varepsilon_1 \cdots \varepsilon_N = -1$$

SO(10)

$\psi_+ :$ $|\varepsilon_1 \, \varepsilon_2 \, \varepsilon_3 \, \varepsilon_4 \, \varepsilon_5\rangle$ (L)

$$\prod_{i=1}^{5} \epsilon_i = +1$$

Lorentz chirality

$|+++++\rangle$ 1

$|--+++\rangle$ $\frac{5!}{2!3!} = 10$

$$1 - - - - + 7 \quad \frac{5!}{1!\,4!} = 5$$

$$\boxed{16 = 10 + \bar{5} + 1}$$

SM: $\begin{pmatrix} u \\ d \end{pmatrix}_L ; \quad (u^c)_L, (d^c)$

$\begin{pmatrix} \nu \\ e \end{pmatrix}_L , \quad (e^c)_L$

$$\begin{pmatrix} d^c \\ \nu \\ e \end{pmatrix}_L \qquad \begin{pmatrix} u^c & u & d \\ & e^c & \end{pmatrix}_L$$

$$\bar{5} \qquad\qquad 10$$

$$\Downarrow \begin{pmatrix} d \\ e^c \\ \nu^c \end{pmatrix}_R = 5$$

$$\psi_{16} = \psi^+ \quad \text{in } SO(10)$$

$$\| $$

$$LH \quad \begin{pmatrix} u \\ d \\ \nu \\ e \\ u^c \\ d^c \\ \nu^c \\ e^c \end{pmatrix}_L$$

$$\boxed{C = \text{element of } SO(10)}$$

$\boxed{\text{Slansky '80}}$

SU(2)

$$\begin{pmatrix} \nu \\ e \end{pmatrix}$$

$\nu \to -\nu$

$e \to -e$

$\boxed{\text{Aulakh, ? 2005}}$

$$\begin{pmatrix} \nu \\ e \end{pmatrix} \to -\begin{pmatrix} \nu \\ e \end{pmatrix}$$

$$U = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} = SU(2)$$

$$= e^{i\pi\sigma_3}$$

$$e_L,\ e_R \iff e_L,\ (e^c)_L$$

LR : $\cancel{f_L \overset{P}{\longleftrightarrow} f_R}$ in $SO(10)$

$$\boxed{f_L \overset{C}{\longleftrightarrow} f_R^* \ (i\gamma_2 f_R^*)}$$

$$SU(2)_L \times SU(2)_R \times \cdots$$

$$\underbrace{g_L = g_R}_{LR} = g \text{ in } SO(10)$$

$$\Downarrow$$

in $SO(10)$ LR symm. is there!

$$\Downarrow$$

it must be C

(NOT P)

$$\cancel{f_L \overset{2}{\longleftrightarrow} f_R}$$

$$\boxed{SO(2) = U(1)}$$

vector $\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \to O \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$

$$O = e^{i\theta L_{12}} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\Leftrightarrow \quad \phi_1 \to \cos\theta\, \phi_1 + \sin\theta\, \phi_2$$

$c \equiv \cos\theta$ $\qquad \phi_2 \to -\sin\theta\, \phi_1 + \cos\theta\, \phi_2$

$s \equiv \sin\theta$

$$\phi_1 - i\phi_2 \to (c\,\phi_1 + s\,\phi_2) - i(-s\,\phi_1 + c\,\phi_2)$$

$$= \phi_1 (c + is) - i\phi_2 (c + is)$$

$$= e^{i\theta} (\phi_1 - i\phi_2)$$

$$\boxed{\phi_1 \pm i\phi_2 \longrightarrow e^{\mp i\theta}\left(\phi_1 \pm i\phi_2\right)}$$

$$\boxed{SO(2) \Leftrightarrow U(1)}$$

$$\Gamma_a = \Gamma_1,\ \Gamma_2 \qquad a = 1,2$$

$$\Gamma_1 = \sigma_1,\ \Gamma_2 = \sigma_2$$

$$\boxed{\Gamma_{FIVE} = (-i)\,\Gamma_1\Gamma_2 = \sigma_3} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$

$$\Sigma_{12} = \frac{1}{4i}\left[\sigma_1, \sigma_2\right] = \frac{\sigma_3}{2}$$

$$[\Gamma_{FIVE}, \Sigma_{12}] = 0$$

$$\boxed{\psi \to e^{i\theta \sigma_3/2} \psi}$$

$$\Rightarrow \boxed{\psi_+ \to e^{i\theta} \psi_+} \quad \boxed{\psi_- \to e^{-i\theta} \psi_-}$$

- $\psi^T B \psi$ $\qquad \Sigma^T B + B \Sigma = 0$

$$\boxed{\sigma_3 B + B \sigma_3 = 0}$$

$\Rightarrow B = \sigma_1$ if $\psi =$ scalar

$B = \sigma_2$ if $\psi =$ fermion

$$\cdot\ \psi^T B \psi = (\psi_+\ \psi_-)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$

$$\propto \psi_+ \psi_-$$

$$\boxed{\text{physics} \therefore \quad \psi_+ = \text{our field}}$$

$$\boxed{\text{NO MASS TERM in } SO(2)}$$

different from Lorentz

$$\left(\psi_L^T C \psi_L = \text{inv.} \right)$$

$$\psi_+ \psi_+ \to \cancel{e^{i\theta}} \psi_+ \psi_+ \quad \text{(not allowed)}$$

- $\psi^T B \Gamma_i \psi \to \psi^T S^T B \Gamma_i S \psi$

$$= \psi^T B \underbrace{S^+ \Gamma_i S}_{O_{ij}\Gamma_j} \psi = O_{ij}\, \psi^T B \Gamma_j \psi$$

$$\Rightarrow \boxed{\psi^T B \Gamma_i \psi \sim \phi_i}$$

$\parallel$

vector of $SO(2) \to SO(2N)$

$$\Rightarrow \boxed{\psi^T B \Gamma_i \psi\, \phi_i = \text{Invariant}}$$

<u>$SO(2)$</u> $\Big\downarrow$ $\Gamma_1 = \sigma_1, \quad \Gamma_2 = \sigma_2$

$$\psi^T \sigma_1 \sigma_1 \psi \, \phi_1 + \psi^T \sigma_1 \sigma_2 \psi \, \phi_2$$

$$= \psi^T \psi \, \phi_1 + i \, \psi^T \sigma_3 \psi \, \phi_2$$

$$= \boxed{\psi_+ \psi_+ \left(\phi_1 + i \phi_2\right)}$$

$$+ \psi_- \psi_- \left(\phi_1 - i \phi_2\right)$$

$$\boxed{\begin{gathered} e^{i\theta} \psi_+ \psi_+ e^{-i\theta} \left(\phi_1 + i \phi_2\right) \\ = i w_1 \end{gathered}}$$

SO(2) = GOOD (CHIRAL)

$\Rightarrow$ NO direct mass

$\Rightarrow$ mass from Yukawa

$\boxed{SO(4)}$ $\Longleftrightarrow$ Lorentz

$\downarrow$ Euclidean $\qquad \updownarrow$ Minkowski

$SO(2)$: $\Gamma_1 = \sigma_1$, $\Gamma_2 = \sigma_2$, $\Gamma_{FIVE} = \sigma_3$

$SO(4)$:

$$\Gamma_a = \begin{pmatrix} 0 & \sigma_a \\ \sigma_a & 0 \end{pmatrix} \qquad a = 1, 2, 3$$

$$\Gamma_4 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \Gamma_i \quad i = \underbrace{1, 2, 3}_{a}, 4$$

$$\Sigma_{12} = \frac{1}{4i}\begin{bmatrix} [\sigma_1, \sigma_2] & 0 \\ 0 & [\sigma_1, \sigma_2] \end{bmatrix}$$

$$= \frac{1}{4i}\, 2i\, \sigma_3 = \sigma_3/2$$

$$\boxed{\Sigma_{ij} = \frac{1}{2}\varepsilon_{ijk}\sigma_k}$$

$i, j = a, b =$
$= 1, 2, 3$

$$\Sigma_{a4} = \frac{1}{2i}\begin{pmatrix} i\sigma_a & 0 \\ 0 & -i\sigma_a \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sigma_a & 0 \\ 0 & -\sigma_a \end{pmatrix}$$

$$\Gamma_{FIVE} = (-i)^2\, \Gamma_1\, \Gamma_2\, \Gamma_3\, \Gamma_4$$

$$= (-1)\begin{pmatrix} \sigma_1\sigma_2 & 0 \\ 0 & \sigma_1\sigma_2 \end{pmatrix}\begin{pmatrix} i\sigma_3 & 0 \\ 0 & -i\sigma_3 \end{pmatrix}$$

$$= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$$\boxed{\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}}$$

- $\psi^T B \psi$ $\qquad$ $B = \Gamma_1 \Gamma_3$

$B = \Gamma_2 \Gamma_4$

$$B = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & \pm i\sigma_2 \end{pmatrix}$$

$$\psi^T B \psi = \underbrace{\psi_+^T i\sigma_2 \psi_+}_{\text{direct mass}} \text{ - - - -}$$

SO(4) = BAD (not chiral)

$\Rightarrow$ mass direct – not from Higgs

$\Downarrow$

- SO(6) = GOOD

  B = off – diagonal

  $\Rightarrow$ no $\psi_+ \psi_+$ $\Rightarrow$ no mass (Higgs)

- $SO(8)$ = BAD

$$\boxed{\bullet\ SO(10) = \text{GOOD} \quad \text{chiral}}$$

$$\psi^T B \psi \Rightarrow \psi_+ \cancel{\psi_-}$$

$$\boxed{\cancel{\psi_+ \psi_+ = \psi_{16} \psi_{16}} \quad SO(10)}$$

<u>not</u> allowed

$SO(4N+2)$ = good

$\cancel{SO(4N)}$ = bad

$SO(12) \Leftarrow$ bad = direct mass