

Neutrino Mass
and
Grand Unification

Lecture XXVIII

8/2/2022

L MU

Winter 2022



SO(10) GUT

1) Neutrino Mass and

My notes: Grand Unification

→ Appendix D = spinors

2) Families from Spinors

Wilczek, Lee 1982

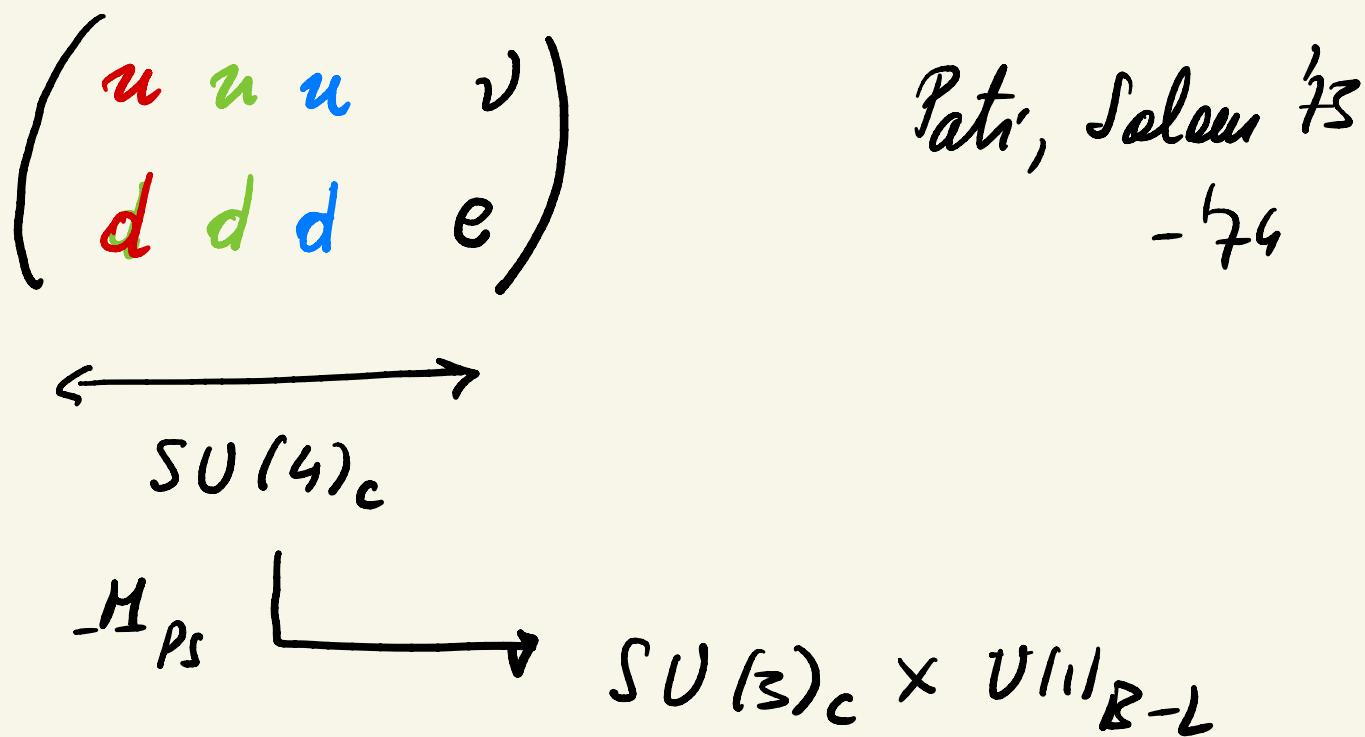
• SM ∴ $G_{SM} = SU(3)_c \times SU(2)_L \times U(1)_Y$

$$SU(5) \supseteq G_{SM}$$

• $L R \therefore$

$$G_{LR} = SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$$

$$SU(4)_c \supseteq SU(3)_c \times U(1)_{B-L}$$



$$(a) SU(2) \times SU(2)'' = "SO(4)"$$

$$\gamma = 2$$

$$\gamma = 2$$

$$m = 3 + 3 = 6$$

$$n = \frac{4 \cdot 3}{2} = 6$$

$$(b) \text{ } SU(4)'' = SO(6)$$

$$r = 3$$

$$r = 3$$

$$n = 4^2 - 1 = 15$$

$$n = \frac{6 \cdot 5}{2} = 15$$

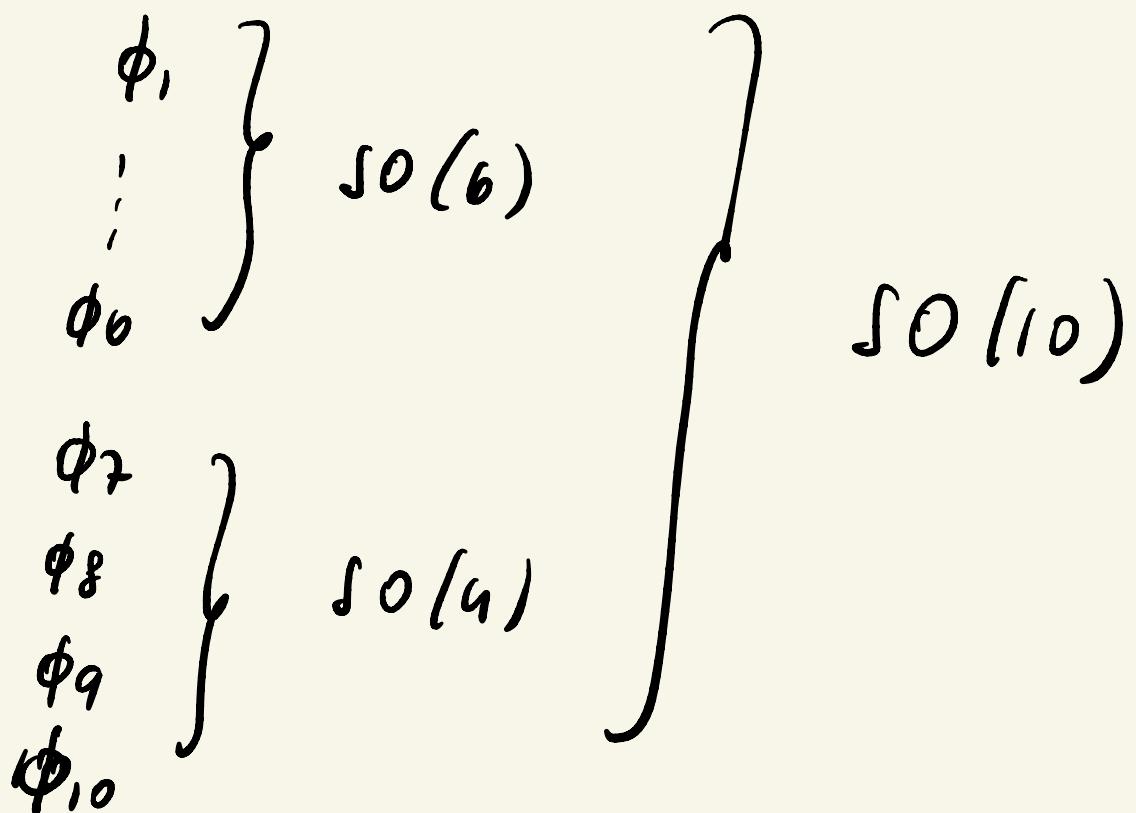
$$\underbrace{SU(2) \times SU(2)}_{\text{"}} \times \underbrace{SU(4)}_{\text{"}} \\ = SO(4) \times SO(6)$$

()

$$\text{Minimal GUT} = SO(10) \supseteq SO(4) \times SO(6)$$



$$\text{Vector of } SO(10) = 10 = \begin{pmatrix} & \phi_1 \\ & \vdots \\ \phi_{10} \end{pmatrix}$$



$$\phi \rightarrow O \phi \quad \therefore OO^T = O^T O = I$$

$$\det O = 1$$

$$O = e^{i \Theta_{ij} L_{ij}}$$

$$(L_{ij})_{ab} = -i (\delta_{ia} \delta_{jb} - \delta_{ib} \delta_{ja})$$

$$L^+ = L, \quad L^T = -L, \quad L^* = -L$$

$$L_{ij} = -L_{ji} \quad n = \frac{2N(2N-1)}{2}$$

$$[L_{ij}, L_{kl}] = i(\delta_{ik} L_{jl} - \delta_{il} L_{jk})$$

$$L_{12} = \begin{pmatrix} 0 & -i & & \\ i & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

$$L_{34} = \begin{pmatrix} 0 & - & 0 & \\ & 0 & -i & \\ 0 & i & 0 & \\ & & & 0 \end{pmatrix}$$

$$[L_{12}, L_{34}] = [L_{12}, L_{56}] = \dots = 0$$

$$C = \{L_{12}, L_{34}, \dots, L_{2N-1, 2N}\}$$

N

$$\gamma(\mathrm{so}(2N)) = N$$

$$V \times V = S + A$$

$\phi \times \phi$ (symmetric) (anti-symmetric)

$$\frac{1}{2} 2N(2N+1)$$

$$\frac{1}{2} 2N \cdot (2N-1)$$

$$SO(10) : A : 10 \cdot 9/2 = 45$$

adjoint

$$S = \frac{10 \cdot 11}{2} = 55$$

$S: \phi_i \phi_j \rightarrow O_{in} \phi_u O_{je} \phi_e$

$S_{ij} \rightarrow O_{in} S_{ue} O_{ej}^T$

$$\boxed{S \rightarrow O S O^T}$$
$$A \rightarrow O A O^T \Rightarrow \text{Tr } A = 0$$

$$\text{Tr } S \rightarrow \text{Tr } O S O^T = \text{Tr } O^T O S = \text{Tr } S$$

S : by subtracting $\text{Tr } S$

$$n_s : \frac{2N(2N+1)}{2} - 1$$

$SO(10)$: $S \rightarrow 5^4$ elements

Spins of $SO(2N)$

Spin $(2N)'' = "SO(2N)"$

$$\left\{ \begin{array}{l} \Gamma_i \quad i = 1, \dots, 2N \\ \{\Gamma_i, \Gamma_j\} = 2\delta_{ij} \end{array} \right.$$

$$\bar{\Sigma}_{ij} = \frac{1}{4i} [\Gamma_i, \Gamma_j]$$

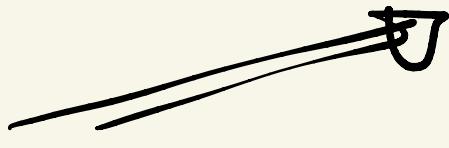
$$[\Sigma_{ij}, \Sigma_{kl}] = i (\delta_{ik} \Sigma_{jl} - \dots)$$

$SO(2N)$ commutation

$\Psi \rightarrow S \Psi$ Spinors

$$S = e^{i \sum_j \theta_{ij} \Sigma_{ij}} \quad \Sigma_{ij}^+ = \Sigma_{ij}$$

$$\Gamma_i = \Gamma_i^+$$



$$\left\{ \begin{array}{l} \Gamma_{FIVE} = (-i)^N \Gamma_1 \Gamma_2 \cdots \Gamma_{2N} (\pm) \\ \therefore \Gamma_{FIVE}^2 = 1 \end{array} \right\}$$

$$\{\Gamma_{FIVE}, \Gamma_i\} = 0$$

$$\Rightarrow [\Gamma_{FIVE}, \Gamma_i \Gamma_j] = \Gamma_{FIVE} \Gamma_i \Gamma_j - \Gamma_i \Gamma_j \Gamma_{FIVE}$$

$$= \Gamma_{FIVE} \Gamma_i \Gamma_j + \Gamma_i \Gamma_{FIVE} \Gamma_j$$

$$- \Gamma_i \Gamma_{FIVE} \Gamma_j - \Gamma_i \Gamma_j \Gamma_{FIVE}$$

$$= \{ \Gamma_{FIVE}, \Gamma_i \} \Gamma_j - \Gamma_i \{ \Gamma_j, \Gamma_{FIVE} \} = 0$$

↓

$$\boxed{[\Gamma_{FIVE}, \Sigma_{ij}] = 0}$$

$$\Rightarrow \Gamma_{\pm} = \frac{1 \pm \Gamma_{FIVE}}{2}$$

$$\boxed{\begin{aligned} \psi_+ &\equiv \Gamma_+ \psi \\ \psi_- &\equiv \Gamma_- \psi \end{aligned}}$$

(L) (R) $\left. \begin{array}{l} \text{Lorentz} \\ \text{Lorentz} \end{array} \right\}$

$$\bullet \quad \Gamma_i' = S^+ \Gamma_i^- S \quad (S^+ S = 1)$$

$$\begin{aligned} \{\Gamma_i', \Gamma_j'\} &= S^+ \{\Gamma_i^-, \Gamma_j^-\} S = \\ &= 2 \delta_{ij} S^+ S = 2 \delta_{ij}' \end{aligned}$$

$$\Rightarrow \Gamma_i' = R_{ij'} \Gamma_j$$

$$\begin{aligned} \{\Gamma_i', \Gamma_j'\} &= R_{iu} R_{je} \{\Gamma_u, \Gamma_e\} \\ &\quad // \\ 2 \delta_{ij} &= 2 R_{iu} R_{ju} \end{aligned}$$

$$\Rightarrow R_{iu} R_{ju} = \delta_{ij}$$

$$(RR^T)_{ij} = \delta_{ij}$$

$$R = 0$$

$$\Gamma'_i = \delta_{ij} R_j$$

old and new
Gammas

Lovely $\psi^T C \psi = \text{Inv.}$

↓
anology

$$\psi^T B \psi = \text{Invariant}$$

↓

$$\psi^T S^T B S \psi = \psi^T B S^+ S \psi = \text{Inv.}$$

$$S^T B = B S^+$$

$$\Sigma^T B = B \Sigma^+$$



$$\Psi^T \Sigma^T B \Sigma \Psi = \Psi^T (1 + iA\Sigma^+) B \\ (1 + i\theta \Sigma) \Psi_{+-}$$

$$= \Psi^T B \Psi + i\theta \Psi^T (\underbrace{\Sigma^T B + B \Sigma}_{\parallel}) \Psi_{+-}$$

$$\boxed{\Sigma_{ij}^T B + B \Sigma_{ij} = 0}$$

$$\Gamma_{FIRE} = (-i)^N \Gamma_1 \dots \Gamma_{2N}$$

$$= 2 \Sigma_{12} 2 \Sigma_{34} \dots 2 \Sigma_{2N-1, 2N}$$

$$2 \Sigma_{ij} = (-1) \Gamma_i \cdot \Gamma_j \quad (i \neq j)$$

$$(2 \Sigma_{ij})^2 = +1$$

$\Sigma_{12}, \Sigma_{34}, \dots : \pm 1$ eigenvalues



denoted as ε_i

$$\Gamma_{FIVE} = 1 \Leftrightarrow |\psi_+\rangle$$

$$= 2 \Sigma_{12} \dots$$

$$= \varepsilon_1, \varepsilon_2 \dots \varepsilon_N$$

$$|\psi_+\rangle = |\varepsilon_1, \varepsilon_2 \dots \varepsilon_N\rangle$$

$\therefore \varepsilon_1, \dots, \varepsilon_N = +1$

$$\psi_- = |\varepsilon_1 \varepsilon_2 \dots \varepsilon_N\rangle$$

$$\therefore \varepsilon_1 \dots \varepsilon_N = -1$$

$SO(10)$

$$\psi_+ : |z, \varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_4 \varepsilon_5\rangle \quad (L)$$

$$\prod_{i=1}^5 \varepsilon_i = +1$$

↑
Lorentz
chirality

$$|+++++\rangle \quad \textcircled{1}$$

$$|---+++ \rangle \quad \frac{5!}{2!3!} = \textcircled{10}$$

$$1 - \dots + > \frac{5!}{1! 4!} = (5)$$

$$16 = 10 + \overline{5} + 1$$

SM: $(u)_L, (u^c)_L, (d^c)_L$

$(\bar{e})_L, (e^c)_L$

$$\left(\begin{array}{c} d^c \\ \bar{e} \end{array} \right)_L \quad \left(\begin{array}{cc} u^c & u^c \\ & e^c \end{array} \right)_L$$

$\overline{5}$ 10

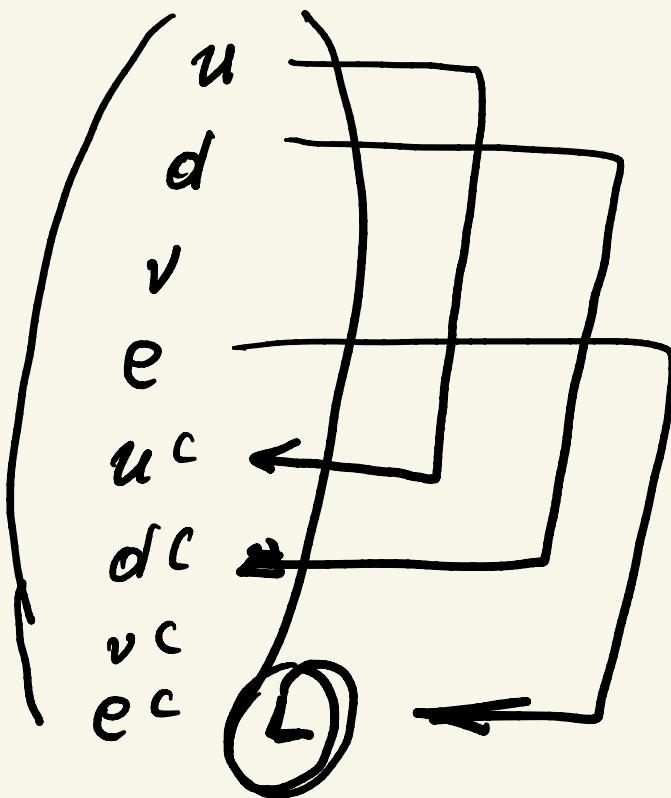


$$\Downarrow \begin{pmatrix} d \\ e^c \\ v_c \\ R \end{pmatrix} = 5$$

- $\gamma_{10} = \gamma^+ \text{ in } SO(10)$

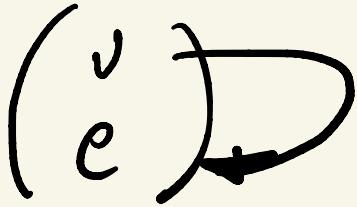
II

LH



C = element of $SO(10)$

$SU(2)$



Slawsky '80

$$v \rightarrow -v$$

$$e \rightarrow -e$$

Aulakh, ?

2005

$$(v) \rightarrow - (v)$$

$$U = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} = SU(2)$$

$$= e^{i\pi \sigma_3}$$

$$e_L, e_R \Leftrightarrow e_L, (e^c)_L$$

$LR :$

$$f_L \xrightarrow{\rho} f_R$$

$\in SO(10)$

$$f_L \xleftarrow{c} f_L^* (i\gamma_2 f_L^*)$$

$SU(2)_L \times SU(2)_R \times - -$

$$g_L = g_R = g_{\text{un } SO(10)}$$

LR



in $SO(10)$ LR sym. is

there !



it must be C

(NOT P)

~~$f_L \xrightarrow{2} f_R$~~

$$SO(2) = U(1)$$

vector $\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \rightarrow O \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$

$$O = e^{i\theta L_{12}} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

$$\Leftrightarrow \phi_1 \rightarrow \cos \theta \phi_1 + \sin \theta \phi_2$$

$$C \equiv \cos \theta \quad \phi_2 \rightarrow -\sin \theta \phi_1 + \cos \theta \phi_2$$

$$S \equiv \sin \theta$$

$$\phi_1 - i\phi_2 \rightarrow (C \phi_1 + S \phi_2) - i(-S \phi_1 + C \phi_2)$$

$$= \phi_1 (C + iS) - i\phi_2 (C + iS)$$

$$= e^{i\theta} (\phi_1 - i\phi_2)$$

$$\phi_1 \pm i\phi_2 \rightarrow e^{\mp i\theta} (\phi_1 \pm i\phi_2)$$

$$SO(2) \Leftrightarrow U(1)$$

$$\cdot \Gamma_a = \Gamma_1, \Gamma_2 \quad a=1,2$$

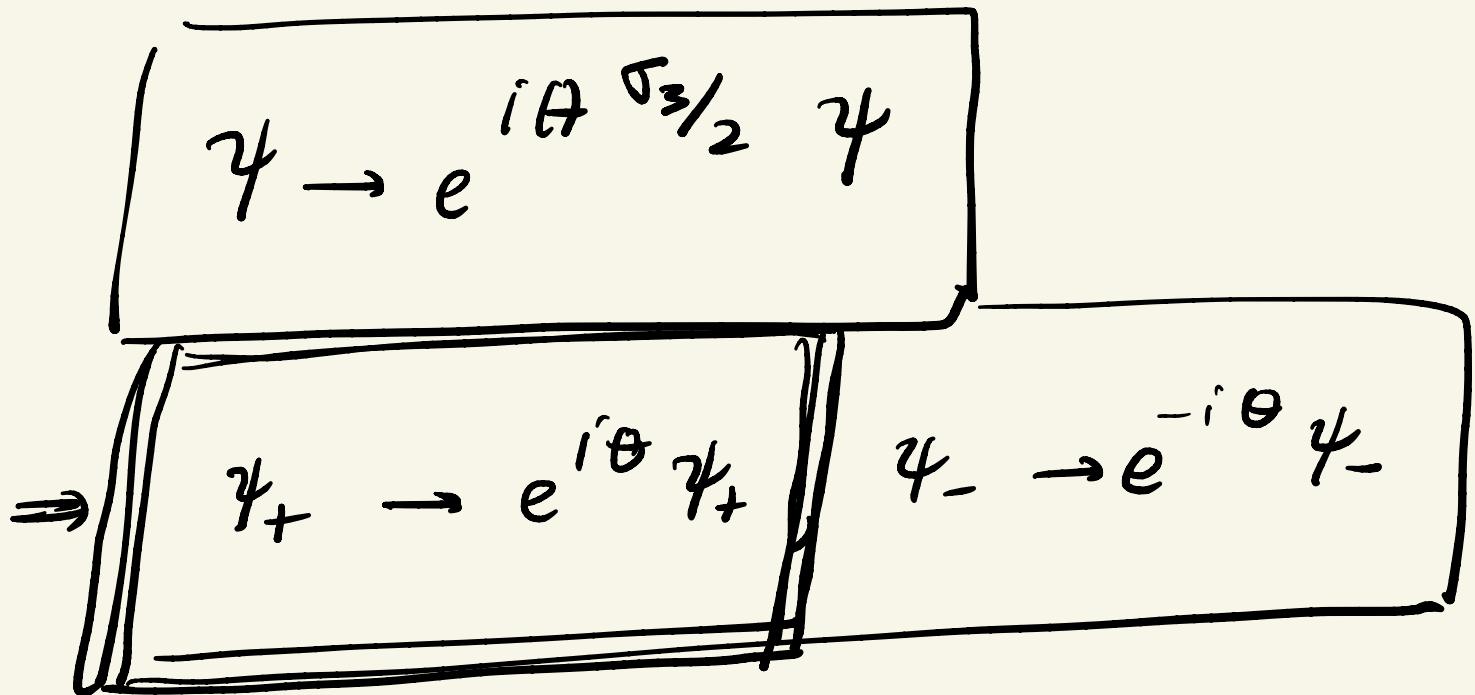
$$\Gamma_1 = \sigma_1, \Gamma_2 = \sigma_2$$

$$\boxed{\Gamma_{FIVE} = (-1) \Gamma_1 \Gamma_2 = \sigma_3 \neq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

$$\gamma = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$

$$\Sigma_{12} = \frac{1}{4\pi} [\sigma_1, \sigma_2] = \frac{\sigma_3}{2}$$

$$[\Gamma_{FIVE}, \Sigma_{12}] = 0$$



- $\psi^T B \psi \quad \Sigma^T B + B \Sigma = 0$

$$\boxed{\sigma_3 B + B \sigma_3 = 0}$$

$$\Rightarrow B = \sigma_1 \quad \text{if } \psi = \text{scalar}$$

$$B = \sigma_2 \quad \text{if } \psi = \text{fermion}$$

$$\cdot \psi^T B \psi = (\psi_+ \psi_-) \begin{pmatrix} 0' \\ , 0 \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$

$$\propto \psi_+ \psi_-$$

physics : ψ_+ = our field

NO MASS TERM

$U(1)$

different from Lorentz

$$(\psi_i^T C \psi_i = i u v.)$$

$$\psi_+ \psi_+ \rightarrow e^{i\theta} \psi_+ \psi_+$$

not allowed

$$\begin{aligned} \cdot \quad & \psi^T B \Gamma_i \psi \rightarrow \psi^T S^T B \Gamma_i \cdot S \psi \\ & = \psi^T B \underbrace{S^+ \Gamma_i \cdot S}_{\partial_{ij} \Gamma_j} \psi = \partial_{ij} \psi^T B \Gamma_j \psi \end{aligned}$$

$$\Rightarrow \boxed{\psi^T B \Gamma_i \psi \sim \phi_i}$$

vector of $SO(2) \rightarrow SO(2N)$

$$\Rightarrow \boxed{\psi^T B \Gamma_i \psi \phi_i = \text{invariant}}$$

$$\begin{array}{c} \hline SO(2) \\ \hline \end{array} \quad \begin{array}{c} \downarrow \\ \Gamma_1 = \sigma_1, \quad \Gamma_2 = \sigma_2 \end{array}$$

$$4^T \sigma_1 \sigma_1 \not\propto \phi_1 + 4^T \sigma_1 \sigma_2 \not\propto \phi_2$$

$$= 4^T \not\propto \phi_1 + i 4^T \sigma_3 \not\propto \phi_2$$

$$= \boxed{4_+ \psi_+ (\phi_1 + i \phi_2)}$$

$$+ \boxed{4_- \psi_- (\bar{\phi}_1 - i \bar{\phi}_2)}$$

$$\textcircled{O} e^{i\theta} \psi_+ \psi_- e^{-i\theta} (\phi_1 + i \phi_2)$$

$$= i w_1$$

$$SO(2) = GOOD (CHIRAL)$$

\Leftrightarrow NO direct mass

\Rightarrow mass from Yukawa

$SO(4)$

\Leftrightarrow Lorentz

Euclidean

Minkowski

$$SO(2): \Gamma_1 = \sigma_1, \Gamma_2 = \sigma_2, \Gamma_{\text{FIVE}} = \sigma_3$$

$$SO(4): \begin{aligned} \Gamma_a &= \begin{pmatrix} 0 & \sigma_a \\ \sigma_a & 0 \end{pmatrix} & a &= 1, 2, 3 \\ \Gamma_4 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & \Gamma_i & \\ && i &= \underbrace{1, 2, 3}_{a}, 4 \end{aligned}$$

$$\Sigma_{12} = \frac{1}{q_i} \begin{bmatrix} [\sigma_1, \sigma_2] & 0 \\ 0 & [\sigma_1, \sigma_2] \end{bmatrix}$$

$$= \frac{1}{q_i} \cdot 2 \cdot \sigma_3 = \sigma_3 / 2$$

$\Sigma_{ij} = \frac{1}{2} \epsilon_{ijk} \sigma_k$

$i, j = a, b =$
 $= 1, 2, 3$

$$\Sigma_{ab} = \frac{1}{2} \begin{pmatrix} i\sigma_a & 0 \\ 0 & -i\sigma_a \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \sigma_a & 0 \\ 0 & -\sigma_a \end{pmatrix}$$

$$\bar{\Gamma}_{FIVE} = (-i)^2 \bar{\Gamma}_1 \bar{\Gamma}_2 \bar{\Gamma}_3 \bar{\Gamma}_4$$

$$= (-i) \begin{pmatrix} \sigma_1 \sigma_2 & 0 \\ 0 & \sigma_1 \sigma_2 \end{pmatrix} \begin{pmatrix} i\sigma_3 & 0 \\ 0 & -i\sigma_3 \end{pmatrix}$$

$$= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$

- $\psi^\top B \psi \quad B = T_1 T_3$

$$B = T_2 T_4$$

$$B = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & \pm i\sigma_2 \end{pmatrix}$$

$$\psi^\top B \psi = \underbrace{\psi_+^\top i\sigma_2 \psi_+}_{\text{direct mass}} \dots$$

$SO(9) = BAD$ (not diagonal)

\Rightarrow mass direct - not

from Higgs

B

• $SO(6) = GOOD$

B = off-diagonal

\Rightarrow no $\psi^+ \psi^- \Rightarrow$ no mass

(Higgs)

• $SO(8) = BAD$

• $SO(10) = GOOD$

chiral

$$\psi^\dagger B \psi \Rightarrow \psi_+^\dagger \cancel{\psi_-}$$

$$\psi_+ \psi_+ = \cancel{\psi_0 \psi_{16}} \quad SO(10)$$

not allowed

$SO(4N+2) = good$

$SO(4N) = bad$

$SO(12)$ \leftarrow bad = direct
mass