Neutrino Mass and Grand Unification

Lecture XXVII

4/2/2022

LMU

Winter 2022

Magnetic Monopoles (3)

Dirac Monopole

Dirac 1931

Charge quantization $\longrightarrow$ magnetic monopoles

(theory)

GUT

'tHooft; Polyakov 1974

example $SO(3) \longrightarrow U(1)$

$\Leftrightarrow \quad SU(5) \to SU(3) \times SU(2)$

$\times U(1)$

$\to SU(3) \times U(1)$

<u>Dirac</u> $\quad \exists$ magnetic monopole

$$\Rightarrow \quad q_e g_m = 2\pi n$$

$$n = 0, \pm 1, \pm 2, \ldots$$

$$\nabla \cdot \vec{B} \propto g_m \, \delta(\vec{r})$$

$\circledast \; g_m \qquad \vec{B} = \frac{g_m}{4\pi} \frac{\hat{r}}{r^2} = \frac{g_m}{4\pi} \frac{\vec{r}}{r^3}$

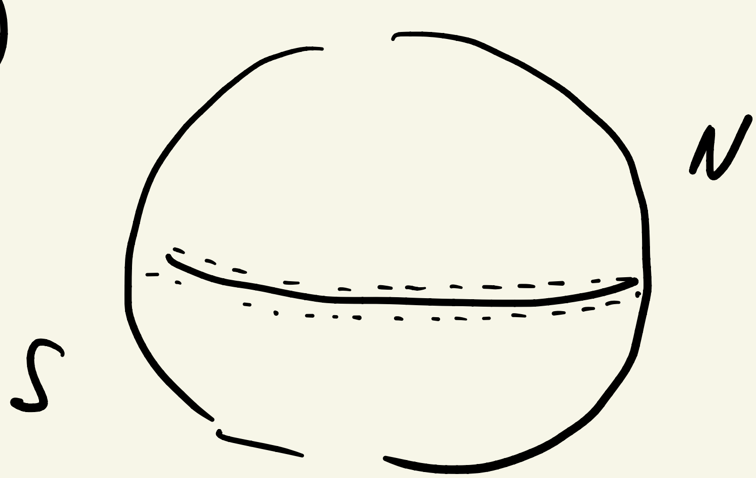
$$\vec{B} \neq \nabla \times \vec{A}$$

$\left(\text{if } \vec{B} = \nabla \times \vec{A} \Rightarrow \nabla \cdot \vec{B} = 0 \right)$

<u>Solution</u>

<u>Wu, Yang ?</u>

(r, θ, φ)



- $0 \leq \theta \leq \pi/2 \quad (N) \quad$ (north)

$$\vec{A}_N = \frac{g_m}{4\pi r} \frac{1-\cos\theta}{\sin\theta} \hat{\varphi}$$

$$1 - \cos\theta \simeq \theta^2/2 + \cdots \quad \Big\{ \quad \frac{1-\cos\theta}{\sin\theta} \xrightarrow[\theta \to 0]{} 0$$

$$\sin\theta \simeq \theta$$

• $\pi/2 \leq \theta \leq \pi$ (S) (south)

$$\vec{A}_S = -\frac{g_m}{4\pi r} \frac{1+\cos\theta}{\sin\theta}\,\hat{\varphi}$$

$$\theta \to \pi \Rightarrow \vec{A}_S \to 0$$

(N) $$\vec{B}_N = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_N \sin\theta\right)\hat{r}$$

$$= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{g_m}{4\pi r}\left(1-\cos\theta\right)\right)\hat{r}$$

$$= \frac{g_m}{4\pi r^2}\,\hat{r}$$ $\leftarrow$ B field of a monopole

$$(S) \quad \vec{B}_S = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_S \sin\theta\right)\hat{r}$$

$$= \frac{g_m}{4\pi r^2}\hat{r} \qquad \boxed{\text{monopole}}$$

$$\boxed{\vec{B}_N = \vec{B}_S = \vec{B}_{mm} = \frac{g_m}{4\pi r^2}\hat{r}}$$

<u>equator</u> $\left(\theta = \pi/2\right)$ $\left(\sin\theta = 1\right)$

$$\boxed{\vec{A}_N - \vec{A}_S = \frac{2 g_m}{4\pi r}\hat{\varphi}}$$

$$= \nabla \chi \qquad (\text{pure gauge})$$

$$\chi = \frac{g_m}{2\pi}\varphi \quad \text{(check)}$$

$$\nabla\chi = \frac{1}{r}\left(\frac{\partial}{\partial\varphi}\chi\right)\hat{r} = \frac{g_m}{2\pi r}\hat{r}$$

But $\boxed{\chi(2\pi) \neq \chi(0) \quad \text{not single-valued}}$

$$\psi_q \to e^{iq\chi}\psi_q \quad \text{(in } e \text{ units)}$$

ⓔ

$$\psi_q \to e^{i\frac{q\,g_m}{2\pi}\varphi}\psi_q$$

$\neq \alpha$

$\uparrow$

gauge transf.

$$e^{i\alpha(2\pi)} = e^{i\alpha(0)} = e^{0} = 1$$

$$\Downarrow$$

$$\boxed{\alpha(2\pi) = 2\pi n}$$

$$\Rightarrow \quad \frac{q\, g_m}{2\pi} \cdot 2\pi = 2\pi n$$

$$\Downarrow$$

$$\boxed{q\, g_m = 2\pi n}$$

Dirac condition

(quantisation)

$$\int_S \vec{B}\cdot d\vec{S} = \int_N \vec{B}_N \, d\vec{S} + \int_S \vec{B}_S \, d\vec{S}$$

$$= \oint \left(\vec{A}_N \, d\vec{l}\,\right) - \oint \vec{A}_S \cdot d\vec{l}$$

$$= \oint \left(\vec{A}_N - \vec{A}_S\right) d\vec{l}$$

$$= \oint (\nabla \chi) \cdot d\vec{l}$$

$$= \frac{g_m}{2\pi} \cdot 2\pi = g_m$$

$$\Longleftrightarrow \quad \vec{B} = \frac{g_m}{4\pi r^2}\,\hat{r}$$

• Dirac $(n=1)$

$$g_m\, q = 2\pi \qquad (\text{in } e \text{ units})$$

e

keep e $\Rightarrow$ $g_m\, q\, e = 2\pi$

$$\boxed{\text{with} \quad g_m\, q\, e = 2\pi n}$$

Maxwell

$\nabla \cdot \vec{E} = \rho_e \qquad \nabla \cdot \vec{B} = 0$

. – – – –

Dirac theory of em with monopole

$\nabla \cdot \vec{E} = \rho_e \qquad \nabla \cdot \vec{B} = \rho_m$

$\vec{B} \neq \nabla \times \vec{A}$

NOT Maxwell

Dirac $(n=1)$

$g_m\, q\, e = 2\pi$

$(q=1) \quad g_m\, e = 2\pi$

't Hooft $\quad SO(3) \qquad g_m\, g = 4\pi$

$\downarrow \qquad \swarrow$

$$\boxed{\vec{B}_m = \frac{\vec{r}}{g r^2}} \Rightarrow \boxed{g = e}$$

$SO(3) = SU(2)$

(A_1, A_2, A_3)

$g = e$

$$\phi_0 = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} \Rightarrow A = A_3$$

$$W_\pm = \frac{A_1 \mp i A_2}{\sqrt{2}}$$

$g = g_W$

$\Downarrow$

$$\boxed{g = e}$$

$$SO(3) \Longrightarrow g_m e = 4\pi$$

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \qquad (T_a)_{ij} = -i\,\varepsilon_{aij}$$

$$\Downarrow$$

$$T_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\phi_0 = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} \Longrightarrow Q = T_3$$

$$Q \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

neutral

$$Q \begin{pmatrix} 1 \\ \pm i \\ 0 \end{pmatrix} = \pm \begin{pmatrix} 1 \\ \pm i \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} (\phi_1 + i\phi_2) &= \phi^+ \\ (\phi_1 - i\phi_2) &= \phi^- \end{aligned}\right\} (Q = \pm 1)$$

$$\boxed{\underline{\text{vector}} \Longleftrightarrow \text{“}s\text{”} = 1}$$

$$SO(3) \longrightarrow SU(2)$$

by “spin” $= 1/2$

$$\psi = \begin{pmatrix} u \\ d \end{pmatrix} \Rightarrow$$

$$Q = T_3 = \sigma_3/2 \text{ on } \psi$$

$$\Rightarrow \pm 1/2 \leftarrow \text{basic charge}$$

$$q_\phi = 2\, q_\psi$$

$$e\, g_m \underbrace{q_\psi}_{(1/2)} = 2\pi$$

$$e\, g_m = 4\pi \Rightarrow e\, g_m\, q_\psi = 2\pi \quad \checkmark$$

$$\boxed{\text{'t Hooft Polyakov} \Longleftrightarrow \text{Dirac}}$$

$\underline{SO(3)}$

$$A_1, \; A_2, \; A_3$$

$$W^\pm \quad (A_1, \; A_2) \; : \; m_W = g v$$

$$A \quad (A_3) \; : \; m_A = 0$$

$$F_{ij} = F_{ij}^{a} \frac{\phi_a}{v} - \cdots -$$

$$\xrightarrow[\infty]{} \underset{\uparrow}{F_{ij}}^{a} \frac{x_a}{r} \quad (\text{monopole})$$

$$Q_{em} = T_a \frac{\phi_a}{v}$$

$$\frac{1}{|\vec{q}|^2} \longleftrightarrow 1/r$$

$$\xrightarrow[\infty]{} \underbrace{T_a \; x_a/r}_{\text{monopole}}$$

$$\frac{1}{|\vec{q}|^2 + m^2} \longleftrightarrow \frac{e^{-mr}}{r}$$

$\wr$

$$\left[\begin{array}{l} W^{\pm} : m_W = 100 \text{ GeV} \\ \iff r_W \simeq 10^{-16} \text{cm} \end{array}\right]$$

GUT monopole

Dokos, Tomaras

$$\Sigma = 24_H \quad \therefore \; g\langle 24\rangle_H = M_X$$

$$\Phi = 5_H \quad \therefore \; g\langle 5_H\rangle = M_W$$

$$Q_{em}(\text{vac}) = \begin{pmatrix} -1/3 & & & & \\ & -1/3 & & & \\ & & -1/3 & & \\ & & & 1 & \\ & & & & 0 \end{pmatrix}$$

$$5_F = \begin{pmatrix} d \\ e^+ \\ \nu^c \end{pmatrix}_R$$

$$5_H = \begin{pmatrix} T \\ \phi^+ \\ \phi^0 \end{pmatrix}$$

$$\Rightarrow Q_{em}(\text{monopole}) = \begin{pmatrix} -1/3 & & & \\ & -1/3 & & \\ & & \boxed{1\cdot\quad} & \\ & & & 0 \end{pmatrix}$$

$$\frac{1}{3} - \frac{2}{3}\,\vec{\sigma}\cdot\hat{r}$$

$$\vec{\sigma}\cdot\hat{r} \to \sigma_3 \ (\text{vacuum}) \Rightarrow \quad \begin{aligned} &1/3 - 2/3 = -1/3 \\ &1/3 + 2/3 = 1 \end{aligned}$$

$$SU(5) \xrightarrow[M_X]{} SU(3)_c \times SU(2)_L \times U(1)_Y$$

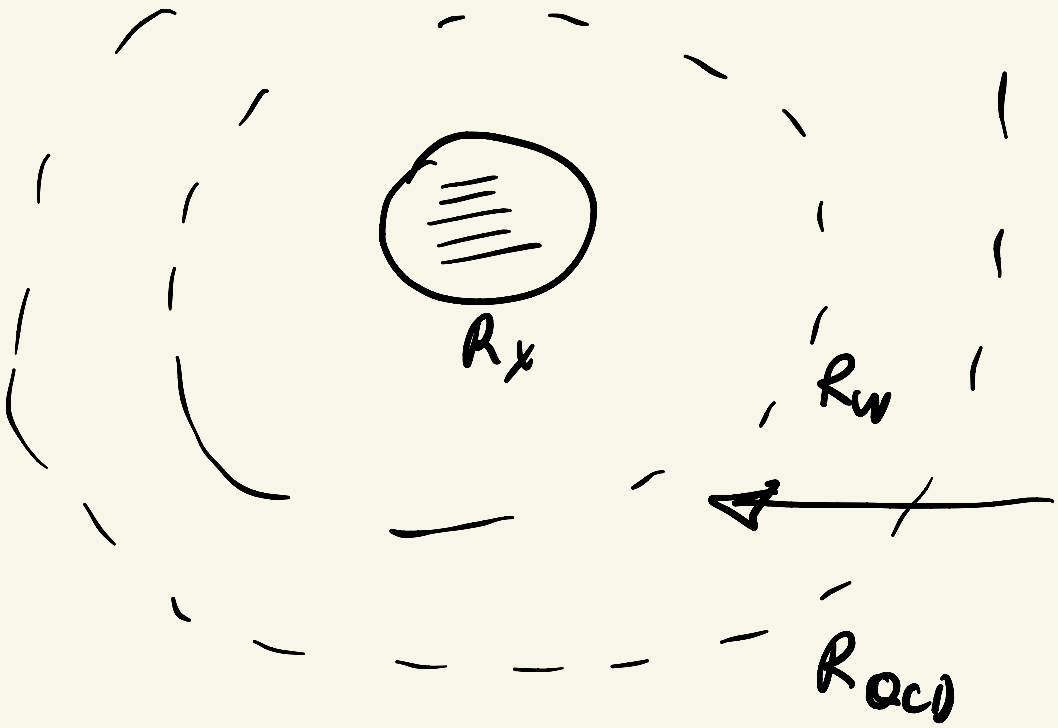
$$\Big\downarrow M_W$$

$$SU(3)_c \times U(1)_{em}$$

$$M_X \longleftrightarrow R_X = 1/M_X \simeq 10^{-30}\ \mathrm{cm}$$

$$M_W \longleftrightarrow R_W = 1/M_W \simeq 10^{-16}\ \mathrm{cm}$$

$$\Lambda_{QCD} \longleftrightarrow R_{QCD} = 1/\Lambda_{QCD} \simeq 10^{-14}\ \mathrm{cm}$$



$$\underline{R \to \infty} \Rightarrow R \gg R_{QCD}$$

$$\Rightarrow \vec{B}_m = \frac{g_m}{4\pi r^2}\hat{r}$$

$$e\, g_m = \frac{2\pi}{Q_q}$$

electron or down quark

$\Downarrow$

$\boxed{\text{turns out}}$

Summary

$$M_m \simeq g_m v \qquad g_m \sim \frac{\pi}{g}$$

$$M_A = g v$$

$$M_h = \lambda v$$

$$m_f = y v$$

Monopole problem

Preskill 1980