Neutrino Mass
and
Grand Unification

Lecture XXVI

1/2/2022

LMU
Winter 2022

Magnetic Monopoles (2)

SO(3) gauge theory

Vector (triplet) Higgs

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \qquad \phi \to O\,\phi$$

$$O O^T = O^T O = 1$$

$$\det O = 1$$

$$O = e^{i\,\theta_a T_a^{(3)}} \qquad a = 1,2,3$$

$$(T_a)_{ij} = -i\,\varepsilon_{aij}$$

$$\phi \iff \Sigma = T_a^{(2)} \phi_a \quad \left(T_a^{(2)} = \frac{\sigma_a}{2} \right)$$

$\uparrow$ $\qquad\qquad\qquad\qquad$ $\uparrow$

vector $\qquad\qquad\qquad$ adjoint of $SU(2)$

$$\mathcal{L} = \frac{1}{2}(D_\mu \phi)^T D^\mu \phi - V(\phi)$$
$$- \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\,\epsilon^{abc} A^b_\mu A^c_\nu$$

$$V(\phi) = \frac{\lambda^2}{2}\left(\phi^T\phi - v^2\right)^2$$

<u>Summary</u>

$$\mathcal{M}_0 = \{\phi_0^T\phi_0 = v^2\} = S_2$$

$$m_{A_1} = m_{A_2} = gv, \quad m_{A_3} = 0$$

$$\phi_0 = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} \qquad Q_{em} = T_3$$

<u>In general</u> $\quad \phi_0{}^a \neq 0$

$$\Rightarrow \bullet \; Q_{em} = T_a \frac{\phi_0{}^a}{v}$$

$$\bullet \; A_\mu = A_\mu{}^a \frac{\phi_0{}^a}{v} \leftarrow \text{(photon)}$$

$$\Downarrow$$

$$\bullet \; F_{\mu\nu} = F_{\mu\nu}{}^a \frac{\phi_0{}^a}{v} - \frac{1}{g} \epsilon_{abc} \times (D_\mu \phi_0)^a (D_\nu \phi_0)^b \phi_0{}^c / v^3$$

(em field)

<u>check</u> $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$; $\phi_0 = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}$

$$Q_{em} = T_3 \Rightarrow$$

$$Q_{em} \phi_3 = 0$$

$$Q_{em} (\phi_1 + i\phi_2) = \pm 1 (\phi_1 + i\phi_2)$$

$$Q_{em} (\phi_1 - i\phi_2) = \mp 1 (\phi_1 - i\phi_2)$$

$$E = \int dV \left[\frac{1}{2} (\vec{B}_a)^2 + \frac{1}{2} (\vec{E}_a)^2 + \right.$$

$\Uparrow$

$$\left. + \frac{1}{2} |D_i \phi|^2 + V(\phi) \right]$$

finite for a static ϕ

$$\left.\begin{array}{l} \Rightarrow \quad V(\phi_m) \to 0 \\ \quad D_i \phi_m \to 0 \\ \quad \vec{B}_a \to 0, \quad \vec{E}_a \to 0 \end{array}\right\} \ (\infty)$$

$$\cdot \ \mathcal{M}_\infty = S_2^\infty \qquad \phi_m(\infty) \leq \mathcal{M}_0$$

$$\phi_m(\infty) : \quad S_2^\infty \longrightarrow S_2^0$$

$$\mathcal{M}_\infty \to \mathcal{M}_0$$

$$\Downarrow$$

$$\boxed{\phi_m^a \Rightarrow v \ \frac{x_a}{r}}$$

(n times)

$$\phi_m^1 \longrightarrow v \sin\theta \cos n\phi$$

$$\phi_m^2 \longrightarrow v \sin\theta \sin n\phi$$

$$\phi_m^3 \to v \cos\theta$$

$$\boxed{n=1} \Rightarrow \boxed{\phi_m = \text{magnetic monopole}}$$

<u>Proof</u>

$$D_i \phi_m \to 0 \qquad \boxed{A_0^a = 0}$$

$$\Rightarrow \boxed{A_i^a \xrightarrow[\infty]{} \pm \frac{1}{g} \epsilon_{aib} \frac{x_b}{r^2}}$$

· $\phi_m = \text{static} \Rightarrow \partial_0 \phi_m = 0$

$$\Rightarrow D_0 \phi_m = 0$$

$$\Rightarrow F_{\mu\nu} \Rightarrow F_{\mu\nu}{}^{a} \, \frac{\phi_{m}{}^{a}}{v} \qquad (\infty)$$

$$\boxed{\begin{array}{c} \cdot\, F_{0i} = \left(\partial_0 A_i^a - \partial_i A_0^a\right) \frac{\phi_m{}^a}{v} = 0 \\ \\ \text{no electric field} \end{array}}$$

$$\cdot\; F_{ij} = F_{ij}{}^{a} \, \frac{\phi_m{}^a}{v}$$

$$\rightarrow \left(\partial_i A_j^a - \partial_j A_i^a + g\,\epsilon_{abc} A_i^b A_j^c\right) \times \frac{x_a}{r}$$

$$A_i^a = \pm \frac{1}{g}\, \epsilon_{aib} \frac{x_b}{r^2}$$

$$F_{ij} \Rightarrow \pm \epsilon_{jab} \left(\frac{\delta_{ib}}{g r^2} + \cancel{\frac{x_b}{g} \partial_i \frac{1}{r^2}} \right) \frac{x_a}{r}$$

$$\mp \epsilon_{iab} \left(\frac{\delta_{jb}}{g r^2} + \cancel{\frac{x_b}{g} \partial_j \frac{1}{r^2}} \right) \frac{x_a}{r}$$

$$+ \cancel{g} \epsilon_{abc} \; \epsilon_{bim} \frac{x_m}{\cancel{g} r^2} \epsilon_{cjn} \frac{x_n}{g r^2} \frac{x_a}{r}$$

$$= \pm \epsilon_{jai} \frac{1}{g r^2} \times 2 \frac{x_a}{r} +$$

$$\frac{1}{g} \left(\cancel{\delta_{aj} \delta_{bn}} - \delta_{an} \delta_{bj} \right) \epsilon_{bim} \frac{x_m x_n x_a}{r^5}$$

$$= \pm \epsilon_{jai} \frac{2 x_a}{g r^3} - \frac{1}{g} \epsilon_{jim} \frac{x_m}{r^3}$$

$$= - \varepsilon_{ija} \frac{2 x_a}{g r^3} + \frac{1}{g} \varepsilon_{ija} \frac{x_a}{r^3}$$

$$= - \varepsilon_{ija} \frac{x_a}{g r^3}$$

$$F_{ij} \equiv \varepsilon_{ija} B_a$$

$$\Rightarrow \boxed{B_a \xrightarrow[\infty]{} - \frac{1}{g} \frac{x_a}{r^3}}$$

- magnetic pole of g_m strength

#

$$\vec{B} = \frac{g_m}{4\pi} \frac{\hat{r}}{r^2}$$

$$B_a = \frac{g_m}{4\pi} \frac{x_a}{r^2}$$

up to a sign !

$$\boxed{g_m = \frac{4\pi}{g}}$$

- $$\phi_m^a(\infty) = v \frac{x_a}{r}$$

$$\vec{\phi}_m(\infty) = v \hat{r}$$

$$\Rightarrow \boxed{\vec{\Phi}_m(0) = 0}$$

maximum of V

(expect) $\quad R_m = \delta \sim \frac{1}{v}$

$$\boxed{\begin{array}{c} SO(3) = \text{toy model} \\ G \to H \simeq U(1) \end{array}}$$

$\mathcal{M}_\infty = S_2, \quad \mathcal{M}_0 = S_2$

$SU(5) \longrightarrow SU(3) \times SU(2) \times U(1)$

$\Downarrow$

$\boxed{\text{magnetic monopoles}}$

$\boxed{\text{Dokos, Tomaras year ?}}$

(Post it)

- $SM \Longrightarrow$ charge quantization of <u>only</u> SM family fermions (anomaly)

$\Downarrow$

· You can add

quarks $\underbrace{Q_L, \quad Q_R}$ same charges

$\Rightarrow$ their charge is arbitrary

$$\Downarrow$$

$$\boxed{\text{NO magnetic monopoles in SM}}$$

$$\underline{SU(2)} \times U(1) \xrightarrow[\langle \Phi \rangle]{} U(1)$$

$\uparrow$ broken

$\hookrightarrow$ doublet

$$\boxed{\text{Is monopole stable?}}$$

<u>YES</u>

$$\hat{\phi}_a \equiv \frac{\phi_a}{v}$$

$$k^\mu = \frac{1}{8\pi}\epsilon^{\mu\nu\alpha\beta}\,\partial_\nu \hat{\phi}_a\,\partial_\alpha \hat{\phi}_b\,\partial_\beta \hat{\phi}_c\,\varepsilon_{abc}$$

$$\partial_\mu k^\mu = 0$$

$$k^0 = \frac{1}{8\pi}\epsilon_{ijk}\,\varepsilon_{abc}\,\partial_i \hat{\phi}_a\,\partial_j \hat{\phi}_b\,\partial_k \hat{\phi}_c$$

$$= \frac{1}{8\pi}\epsilon_{ijk}\,\varepsilon_{abc}\Big[\partial_i\left(\hat{\phi}_a\,\partial_j \hat{\phi}_b\,\partial_k \hat{\phi}_c\right) - \cancel{\hat{\phi}_a\,\partial_i\partial_j \,--}\Big]$$

$$\Downarrow$$

$$k^0 = \frac{1}{8\pi} \partial_i V_i$$

monopole $V_i = \varepsilon_{ijk} \varepsilon_{abc} \hat{\phi}_a^m \partial_j \hat{\phi}_b^m \partial_k \hat{\phi}_c^m$

$$\Downarrow$$

$$Q_m = \int k^0 d^3x = \int_\infty dS_i \, V_i \, \frac{1}{8\pi}$$

$$V_i(\phi_m) \to \varepsilon_{ijk} \varepsilon_{abc} \frac{x_a}{r} \partial_j \left(\frac{x_b}{r}\right) \partial_k \left(\frac{x_c}{r}\right)$$

$$= \varepsilon_{ijk} \varepsilon_{abc} \frac{x_a}{r} \frac{\delta_{jb}}{r} \frac{\delta_{kc}}{r} =$$

$$= \varepsilon_{abc} \varepsilon_{ibc} \frac{x_a}{r^3} = 2 \delta_{ai} \frac{x_a}{r^3} = 2 \frac{x_i}{r^3}$$

$$\Downarrow$$

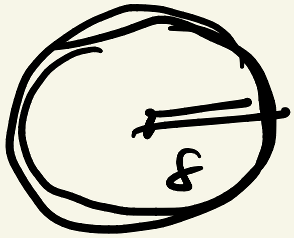
$$Q_m = \int dS_i \, \frac{1}{4\pi} \frac{x_i}{r^3} =$$

$$= \int_{r\to\infty} dS \, \hat{r} \, \frac{1}{4\pi} \frac{\hat{r}}{r^2} = 4\pi \int \frac{1}{4\pi} \, d\Omega = 1$$

$$\boxed{\begin{aligned} Q_m(\phi_m) &= 1 \\ Q_m(\phi_0) &= 0 \end{aligned}}$$

$(dS = r^2 d\Omega)$

$$\boxed{\frac{dQ}{dt} = 0 \Rightarrow \text{monopole is stable}}$$



monopole

inside $R=\delta$

$\phi_m = 0, \ \vec{B} = 0$

outside : $\vec{\phi}_m = v\hat{r}$

$\vec{B} = \frac{1}{g r^2}\hat{r}$

$$E = \int d^3x \left[\frac{1}{2}\vec{B}^2 + V \right]$$

$$= \int_0^\delta d^3x \underbrace{V}_{=\frac{\lambda^2}{2}v^4} + \int_\delta^\infty d^3x \frac{1}{2}\vec{B}^2$$

$\Downarrow$

minimizing $E(\delta)$ ...

$$\frac{\partial E(\delta)}{\partial \delta} = 0$$

$$\Rightarrow \boxed{\delta \simeq \frac{1}{v} \frac{1}{\sqrt{g\lambda}}}$$

<u>Friday</u>

$\exists$ magnetic monopole

$\Downarrow$ $\boxed{\text{Dirac}}$

$$\boxed{g_m \, q_e = 2\pi n \qquad n = \pm 1, \ldots}$$

Cosmology

- $R_U \simeq \dfrac{10^{30}}{T} \qquad d_H \simeq \dfrac{M_{pl}}{T^2}$

$\boxed{\text{Imagine 1 monopole per horizon } (T \simeq M_X)}$

$\Rightarrow$ compute the E_m and compare with E_U

$E_U \simeq 10^{80}\ \mathrm{GeV}$

$$M_m \simeq \frac{1}{g\,v_X} \simeq \frac{1}{g^2} M_X \simeq 10^{17}\ \mathrm{GeV}$$

$$M_x \simeq g\,\vartheta_x$$