Ludwig-Maximilians-Universität München

Quantum Field Theory (Quantum Electrodynamics)

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12 February 2020

First and last name : ______________________

Matriculation number : ______________________

Number of extra sheets : ______________________

Guidelines :

- The exam consists of 4 problems.
- The duration of the exam is 3 hours.
- Please write your name and matriculation number on every sheet that you hand in. State the number of extra sheets.
- You are not allowed to use books or notes. Some potentially useful formulas can be found on the last page.
- Do you agree that your results be published on the course website ? ☐ yes ☐ no
- Your answers should be comprehensible and readable.

GOOD LUCK !

Do not write below this line.

Comments :

Exercise 1	/ 16 P
Exercise 2	/ 40 P
Exercise 3	/ 22 P
Exercise 4	/ 22 P

Total	/ 100 P
Grade	

Problem 1 (16 points)

Consider a column of N real scalar fields

$$\Phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \\ \vdots \\ \phi_N(x) \end{pmatrix}$$

1. Construct the most general Lagrangian (in four spacetime dimensions) which is Lorentz and $O(N)$ invariant and contains terms with mass dimension at most 4 in Φ and its derivatives.

   Hint : The $O(N)$ transformation acts on the fields as $\phi'_i = \sum_j O_{ij}\phi_j$, with O_{ij} a (constant) real orthogonal matrix ($O^T O = 1$).
2. Find the equations of motion.
3. In its infinitesimal form, an $O(N)$ transformation can be written as

   $$O_{ij} = \delta_{ij} + \sum_A \epsilon_A T^A_{ij} + \mathcal{O}(\epsilon^2) \ ,$$

   where $\epsilon^A \ll 1$ are the (constant) parameters of the transformation, T^A_{ij} are the so-called generators of $O(N)$, and A runs over the number of independent generators. Show that the T^A_{ij} are antisymmetric matrices.
4. Find the Noether current associated with the $O(N)$ invariance of the theory. Show that it is conserved on the equations of motion.
5. Derive the Hamiltonian of this theory.

Problem 2 (40 points)

Consider the following Lagrangian (in four spacetime dimensions)

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}M^2\phi^2 + \bar{\psi}(i\partial\!\!\!/ - m)\psi - g\phi\bar{\psi}\gamma_5\psi \ .$$

Here ϕ is a real scalar field with mass M, ψ the electron-positron field with mass m, g a coupling constant, and $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$.

1. Check if the Lagrangian is invariant under $\psi \to e^{i\alpha}\psi$, with α a constant. If so, derive the corresponding Noether current.
2. Consider the process

   $$e^- e^+ \to e^- e^+ \ .$$

   Draw and label the leading-order Feynman diagram or diagrams. Please use p_1, p_2 to indicate the incoming momenta, and p_3, p_4 the outgoing momenta.
3. Derive the spin-averaged amplitude squared in terms of the Mandelstam variables, in the limit $m \to 0$.

4. Using the above result, derive the spin-averaged amplitude squared for the process

$$e^- e^- \to e^- e^- \ .$$

Hint : You should not need to do any explicit computation.

5. Consider the process $\phi \to e^+ e^-$. For what masses can this process take place?

Problem 3 (22 points)

1. Write down the QED Lagrangian.
2. Derive the equations of motion for the fields.
3. Write down the Feynman diagrams for the process $e^- \gamma \to e^- \gamma$ and show that the corresponding amplitude can be written as

$$i\mathcal{M} = \epsilon^\mu(\vec{k}_1)\epsilon^\nu(\vec{k}_2)\mathcal{A}_{\mu\nu} \ .$$

Hint : You don't need to fully simplify your result.

4. Show that $k_1^\mu \mathcal{A}_{\mu\nu} = 0$ and explain why this is expected.
Hint : Use four-momentum conservation and Dirac equation in momentum space.

Problem 4 (22 points)

1. Using a complex scalar field ϕ with mass m_ϕ and the photon A_μ, write down the most general Lorentz and $U(1)$ gauge invariant Lagrangian up to (and including) terms of mass dimension 4.
2. Draw and label the two one-loop Feynman diagrams associated with the photon vacuum polarization in this theory.
3. Now work in four spacetime dimensions and check that $q^\mu \Pi_{\mu\nu}(q) = 0$.
Hint : You do not need to explicitly carry out the integrals. Use the relation $\pm 2qk + q^2 = (k \pm q)^2 - m^2 - (k^2 - m^2)$.

Potentially useful formulas

Decomposition in terms of ladder operators

Massive real scalar field

$$\phi(x) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} \left(a(\vec{p}) e^{-ipx} + a^+(\vec{p}) e^{ipx} \right) \ , \quad \omega_{\vec{p}} = \sqrt{|\vec{p}|^2 + m^2} \ .$$

$$[a(\vec{p}), a^+(\vec{p'})] = (2\pi)^3 2\omega_{\vec{p}}\, \delta^{(3)}(\vec{p} - \vec{p'}) \ .$$

Massive complex scalar field

$$\chi(x) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} \left(a(\vec{p}) e^{-ipx} + b^+(\vec{p}) e^{ipx} \right) \ , \quad \omega_{\vec{p}} = \sqrt{|\vec{p}|^2 + m^2} \ .$$

$$[a(\vec{p}), a^+(\vec{p'})] = [b(\vec{p}), b^+(\vec{p'})] = (2\pi)^3 2\omega_{\vec{p}}\, \delta^{(3)}(\vec{p} - \vec{p'}) \ .$$

Massive fermionic field

$$\psi(x) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} \sum_i \left(u_i(\vec{p}) a_i(\vec{p}) e^{-ipx} + v_i(\vec{p}) b_i^+(\vec{p}) e^{ipx} \right) \ , \quad \omega_{\vec{p}} = \sqrt{|\vec{p}|^2 + m^2} \ .$$

$$\{a_i(\vec{p}), a_j^+(\vec{p'})\} = \{b_i(\vec{p}), b_j^+(\vec{p'})\} = (2\pi)^3 2\omega_{\vec{p}} \delta_{ij}\, \delta^{(3)}(\vec{p} - \vec{p'}) \ .$$

$$(\not{p} - m) u(p) = 0 \ , \quad \bar{u}(p)(\not{p} - m) = 0 \ .$$

$$(\not{p} + m) v(p) = 0 \ , \quad \bar{v}(p)(\not{p} + m) = 0 \ .$$

$$\sum_s u_s(p) \bar{u}_s(p) = \not{p} + m \ , \quad \sum_s v_s(p) \bar{v}_s(p) = \not{p} - m \ .$$

Photon

$$A^\mu(x) = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}} \sum_{r=0}^{3} \epsilon_r^\mu \left(a_r(\vec{k}) e^{-ikx} + a_r^+(\vec{k}) e^{ikx} \right) \ , \quad \omega_{\vec{k}} = |\vec{k}| \ .$$

$$[a_r(\vec{k}), a_s^+(\vec{k'})] = (2\pi)^2 2\omega_{\vec{k}} \zeta_s \delta_{rs} \delta^{(3)}(\vec{k} - \vec{k'}) \ , \quad \zeta_0 = -1, \zeta_{1,2,3} = 1 \ .$$

$$\eta_{\mu\nu} \epsilon_r^\mu \epsilon_s^\nu = -\zeta_s \delta_{rs}, \quad \sum_{r=0}^{3} \zeta_r \epsilon_r^\mu \epsilon_r^\nu = -\eta^{\mu\nu} \ .$$

γ matrices properties

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \ , \quad \gamma^\mu\gamma_\mu = 4, \quad \gamma^\mu\gamma^\alpha\gamma_\mu = -2\gamma^\alpha, \quad \gamma^\mu\gamma^\alpha\gamma^\beta\gamma_\mu = 4\eta^{\alpha\beta},$$
$$\gamma^\mu\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma_\mu = -2\gamma^\gamma\gamma^\beta\gamma^\alpha,$$
$$(\gamma_5)^2 = 1, \quad \gamma_5^+ = \gamma_5, \quad \text{with} \quad \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 \ ,$$
$$\{\gamma_5, \gamma_\mu\} = 0 \ ,$$
$$\mathrm{tr}(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu} \ ,$$
$$\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta) = 4(\eta^{\mu\nu}\eta^{\alpha\beta} + \eta^{\mu\beta}\eta^{\nu\alpha} - \eta^{\mu\alpha}\eta^{\nu\beta}) \ .$$
$$\mathrm{tr}(\gamma_5) = 0 \ .$$

Projection Operators

$$P_{L,R} = \frac{1 \mp \gamma_5}{2},$$

$$P_L^2 = P_L \ , \quad P_R^2 = P_R \ , \quad P_L P_R = 0 \ , \quad P_R P_L = 0 \ , \quad P_L + P_R = 1$$

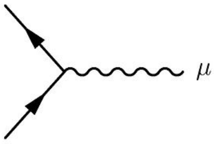
Mandelstam variables

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2 \ , \quad t = (p_1 - p_3)^2 = (p_2 - p_4)^2 \ , \quad u = (p_1 - p_4)^2 = (p_2 - p_3)^2 \ .$$

Feynman rules for QED

 $\rightarrow \left(\frac{i}{\not{p} - m + i\varepsilon}\right)$

$\rightarrow \frac{-i\eta_{\mu\nu}}{p^2 + i\varepsilon}$



$\rightarrow -ie\gamma^{\mu}$

Incoming fermion: $\rightarrow u\ (\vec{p}, s)$

Incoming antifermion: $\rightarrow \overline{v}\ (\vec{p}, s)$

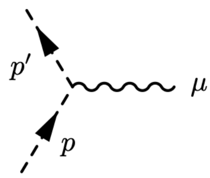
Outgoing fermion:  $\rightarrow \overline{u}\ (\vec{p}, s)$

Outgoing antifermion: $\rightarrow v\ (p, s)$

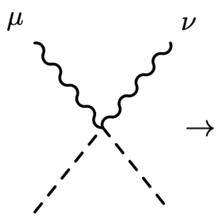
photon: 

$\rightarrow \epsilon_{\mu}(\vec{k}, \lambda)$

Cubic and quartic vertices for scalar - photon interaction



$\rightarrow ie(p + p')^{\mu}$



$\rightarrow 2ie^2\eta^{\mu\nu}$