LMU GUT Course

Lecture VII

24/11/2020

LMU
Fall 2020

Strings

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^* (D^\mu \phi)$$

$$- V(|\phi|) \qquad (1)$$

$$\phi \in \mathbb{C} \qquad \phi \to U \phi$$

$$U = e^{i\alpha Q} \qquad (Q\phi = 1) \quad (2)$$

$$D_\mu = \partial_\mu - ig A_\mu \qquad (g \neq e)$$

$$V = \frac{\lambda}{4} \left(|\phi|^2 - v^2 \right)^2 \qquad (3)$$

$$\Downarrow$$

$$\mathcal{M}_0 = \{ \phi_0 \therefore V = V_{min} \}$$

$$= \{ \phi_0 \therefore V = 0 \}$$

$$= \{ \phi_0 \therefore |\phi_0|^2 = v^2 \}$$

$$= S_1$$

$$U(1) \longleftrightarrow \phi \to U\phi$$

$$\|$$

$$SO(2) \qquad \phi = \phi_1 + i\phi_2$$

$$\phi \to e^{i\alpha}\phi$$

$$\Rightarrow \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

$$\boxed{(g \neq e)}$$

$$\| \quad O : O^T O = 1, \quad \det O = 1$$

- $\phi_0 = v$ vacuum

$$\boxed{\text{static, finite energy classical solution}}$$

$$E = \int d^3x \left[V + |D_i \phi|^2 + \frac{1}{2} \vec{B}^2 \right]$$

$$\boxed{\to 0 \text{ at } \infty}$$

(4)

$\phi_{cl}(\infty) \longrightarrow \mathcal{M}_0$

$\mathcal{M}_\infty \longrightarrow \mathcal{M}_0$

$\Downarrow$

$\boxed{\mathcal{M}_\infty = S_1}$

S_1 at ∞

$\longleftarrow \boxed{\text{lmg}}$

Nielsen - Olesen '71

$$|\phi^{cl}_\infty|^2 \longrightarrow v^2$$

- $\underbrace{\phi^{cl}_\infty = V = \phi_0}_{\text{vacuum}}$

- $\phi_{cl}^\infty = v e^{i\theta(n)}$

$$\boxed{S_1 \longrightarrow S_1}$$

$$\Downarrow$$

$$D_i \phi_\infty \longrightarrow 0$$

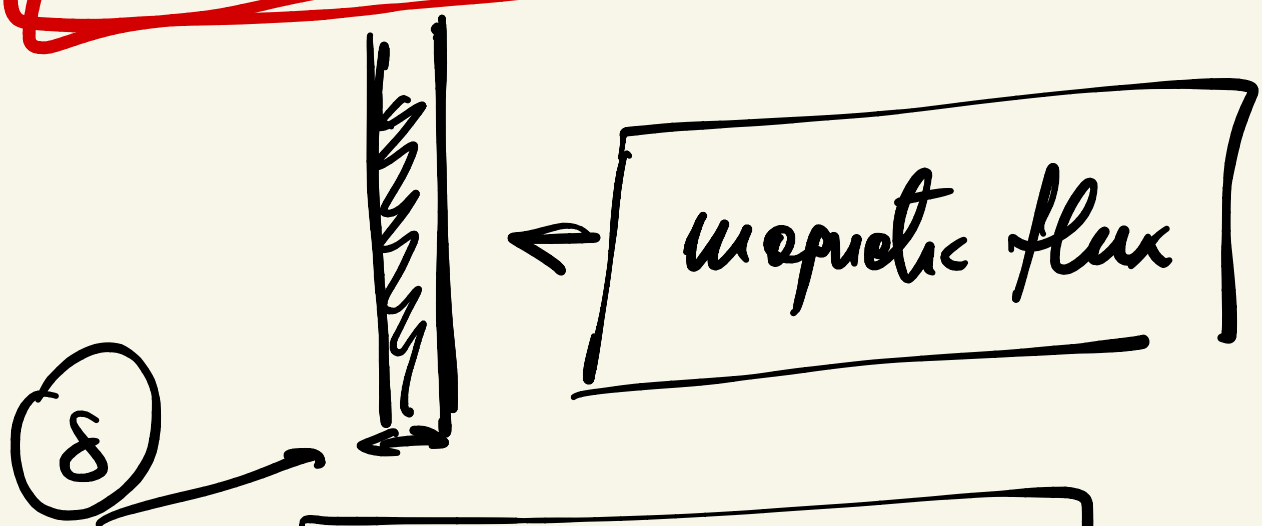
$$\rightarrow (\partial_i - ig A_i^\infty) v e^{i\theta} = 0$$

$$\Rightarrow \boxed{A_i^a = \frac{(n)}{g}\,\partial_i \Theta} \qquad (5)$$

$$\oint dx^i A_i^a = \frac{(n)}{g} \underbrace{\oint dx_i\, \partial_i \Theta}_{} = \Delta\Theta$$

$$\parallel \qquad\qquad = \frac{(n)}{g}\, 2\pi$$

$$\int d\vec{S}\, \vec{B} = \Phi_B \rightarrow$$



$$\boxed{E/L = \text{finite}}$$

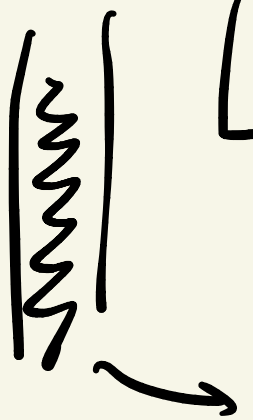
$$L \to \infty \Rightarrow E \to \infty$$

$$\delta \sim \frac{1}{v}$$

$$\boxed{\Phi_B = \frac{2\pi}{g}(n)}$$

$$E/c = \int d^2x \left[V + |D_i\phi|^2 + \tfrac{1}{2}\vec{B}^2 \right]$$

$$\boxed{\delta \to 0} \qquad \Longrightarrow \boxed{\delta \to \infty}$$



δ

$$\Phi_B = B \cdot \delta^2$$

$$B \sim \frac{1}{\delta^2}$$

$$\boxed{\begin{array}{l} \text{Inside:} \quad \phi_{cl} \xrightarrow[\rho \to 0]{} 0 \\ \\ (x, y) \longrightarrow (\rho, \theta) \end{array}}$$

$$V = \lambda\left(|\phi|^2 - v^2\right)^2$$

$$\left(E/L\right) = \underbrace{v^4 \delta^2}_{V} + \frac{1}{\delta^4}\cdot\delta^2$$

$$\frac{\partial}{\partial\delta}\left(E/L\right) = 0 \rightarrow v^4\delta \simeq \frac{1}{\delta^3}$$

$$\Rightarrow \boxed{\delta \sim \frac{1}{v}}$$

$$\boxed{\left(M/L\right)_m = \left(E/L\right) \gg R_c^{-1}}$$

$$\| \quad \frac{1}{g} v$$

$$\boxed{R_c \sim g v}$$

- monopole $\quad SU(2) \to U(1)$

$$\mathcal{M}_\infty = S_2 \longrightarrow \mathcal{M}_0 = S_2$$

- string $\quad U(1) \to \mathbb{1}$

$$\mathcal{M}_\infty = S_1 \longrightarrow \mathcal{M}_0 = S_1$$

- $\mathcal{M}_\infty = \{\text{few points}\} \longrightarrow \mathcal{M}_0 = \{\text{few points}\}$

few points $\longrightarrow$ $\boxed{n = 2 \text{ points}}$

$\cancel{P}$, $\cancel{C}$, $\cancel{CP}$ (discrete Z_2)

$$\phi \xrightarrow{D} -\phi \qquad \boxed{\phi \in \mathbb{R}}$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi)$$

$$V = \frac{\lambda}{4}(\phi^2 - v^2)^2$$

$$\mathcal{M}_0 = \{ \phi_0 : \phi_0^2 = v^2 \}$$

$$= \{ \phi_0 = +v, -v \}$$

$$\mathcal{M}_\infty = \{ z = +\infty, -\infty \}$$

- $\phi_{cl} = v$ for all z

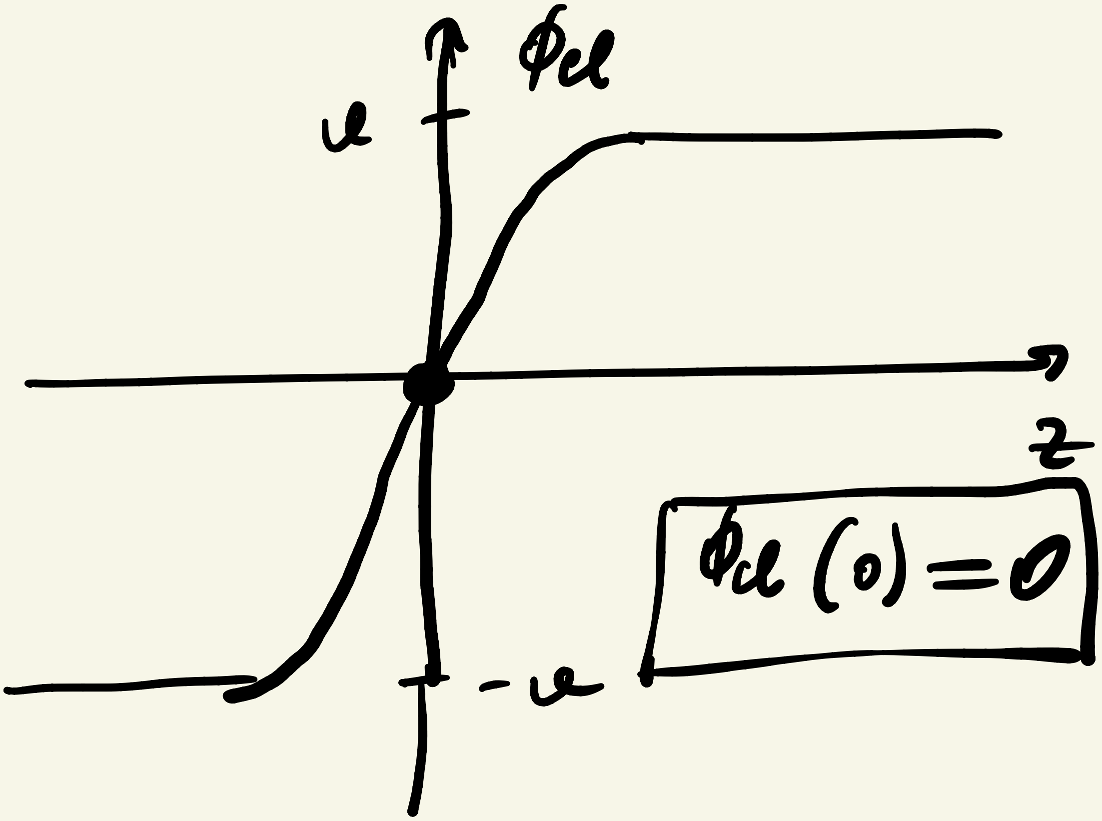
vacuum $\phi_{cl} = \phi_0$

trivial maps

- $\phi_{cl}(+\infty) = +v$

$\phi_{cl}(-\infty) = -v$

$\phi_{cl}(z)$



$$\Box \phi = -\frac{\partial V}{\partial \phi} \qquad (7)$$

$$-\frac{d^2 \phi_{cl}}{dz^2} = -\frac{\partial V}{\partial \phi} \Big/ \frac{d\phi}{dz} \qquad (8)$$

$$\Rightarrow \frac{d}{dz}\,\frac{1}{2}\left(\frac{d\phi_{cl}}{dz}\right)^2 = \frac{dV}{dz} \qquad (9)$$

$$\Downarrow$$

$$\frac{1}{2}\left(\frac{d\phi_{cl}}{dz}\right)^2 = V + \text{const.}\ (C) \qquad (10)$$

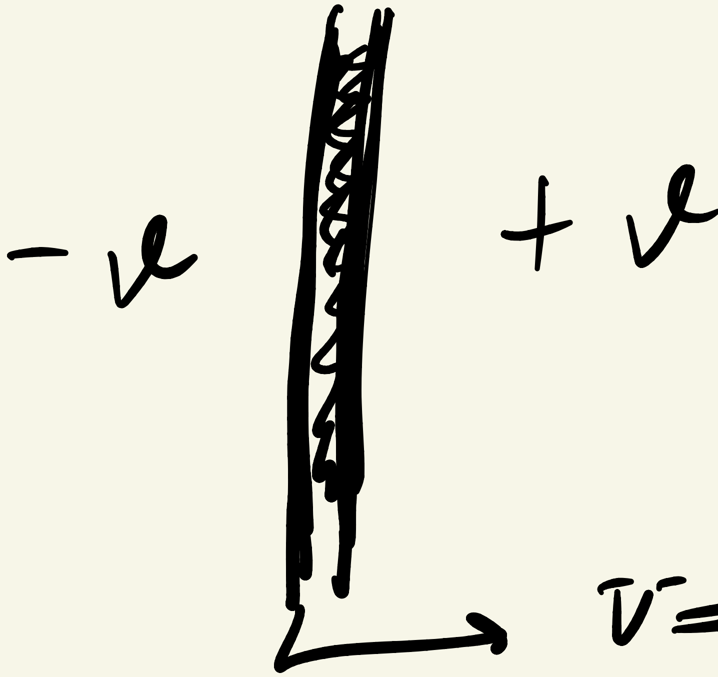
$$\boxed{C = ?} \qquad \left(\frac{d\phi_{cl}}{dz}\right)^2 \xrightarrow{\infty} 0 \qquad V \xrightarrow{\infty} 0$$

$$\Rightarrow \boxed{C = 0}$$

$$\Downarrow$$

$$\frac{d\phi_{cl}}{dz} = \pm\sqrt{2V}$$

$$= \pm\frac{\sqrt{\lambda}}{2}\left(\phi^2 - a^2\right)$$



$$\frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 \longleftrightarrow V = \frac{\lambda}{4} v^4$$

$$\boxed{\phi_{cl} = \pm v \tanh \sqrt{\frac{\lambda}{2}}\, z v}$$

Exercise

$$\boxed{M_m \gg \frac{1}{R_0}} \qquad \boxed{**}$$

Road to unification

$\underline{SU(2)} \longrightarrow U(1)$

unified ew theory

$\Rightarrow$ failed

$SU(3)_c$ of strong int.

$\Downarrow$

$$\boxed{G_{min} = SU(5)} \quad (r = 4)$$

$\Downarrow$ |

$SU(3) \times SU(2) \times U(1)$

$r = 2 + 1 + 1 = 4$

$\parallel \qquad \Downarrow$

GUT = Grand Unified Theory

$$\boxed{M_{GUT} = 10^{16}\ \text{GeV}}$$

$$\boxed{\text{E-W unification?}}$$

Weinberg '70s

Lee

$$\underbrace{e_W = SU(2)_L \times U(1)_Y}_{SU(3)}$$

minimal gauge group

SU(3)

$$U_3 U_3^\dagger = 1, \quad \det U_3 = 1$$

$$\Downarrow$$

$$U = e^{iH} \qquad \boxed{H = H^\dagger}$$

$$\det U = 1 \Rightarrow \boxed{\mathrm{tr}\, H = 0}$$

$$H = T_a \theta_a \qquad 3 \times 3 = 8 + 1$$

$$T_a \equiv \frac{\lambda_a}{2} \qquad a = 1, \dots, 8$$

Gell-Mann

$$\boxed{\mathrm{Tr}\, T_a T_b = \frac{1}{2} \delta_{ab}}$$

$$\Rightarrow \mathrm{Tr}\, \lambda_a \lambda_b = 2 \delta_{ab}$$

$$\lambda_1 = \begin{pmatrix} \overset{=\sigma_1}{\boxed{\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} \overset{=\sigma_2}{\boxed{\begin{matrix} 0 & -i \\ i & 0 \end{matrix}}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

Cartan

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\boxed{[T_a, T_b] = i f_{abc} T_c}$$

$$F \to U F \quad \text{(fundamental)}$$

$$\boxed{\bar F \to \bar F U^\dagger} \quad \text{(anti-fund)}$$

$$F = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \to U \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

$$\bar F^T = \begin{pmatrix} \bar f_1 \\ \bar f_2 \\ \bar f_3 \end{pmatrix} \to U^* \begin{pmatrix} \bar f_1 \\ \bar f_2 \\ \bar f_3 \end{pmatrix}$$

$$\bar f_i \sim f_i^*$$

- $\underbrace{F \times \bar F}_{\Sigma} \to U F \times \bar F U^\dagger$

Σ = adjoint repn

$$\Sigma \to U \Sigma U^\dagger$$

$$\Rightarrow \operatorname{Tr} \Sigma = 0, \quad \Sigma = \Sigma^\dagger \quad \text{(irred)}$$

$\boxed{d=8}$ $\quad \Sigma = T_a \varphi_a \qquad a = 1, \ldots, 8$

- $F_i \times F_j \to U_{ik} U_{jl} F_k F_l$

$$= U_{ik} \underbrace{F_k F_l} U_{lj}^T$$

$$\boxed{F \times F \to U \, F \times F \, U^T}$$

$\Rightarrow \quad S \to U S U^T \qquad S^T = S$ $\quad \boxed{d=6}$

symmetric

$A \Rightarrow U A U^T \qquad A^T = -A$

anti-symmetric $\quad \boxed{d=3}$

- $3 \times \bar{3} = 8 + 1$

• $3 \times 3 = 6 + 3^*$ (check)

$$\underline{\varepsilon_{ijk} F_i F_j F_k} \longrightarrow$$

$$\rightarrow \varepsilon_{ijk} \, U_{ii'} U_{jj'} U_{kk'} \, F_{i'} F_{j'} F_{k'}$$

$$= (\underbrace{\det U}_{=1}) \, \varepsilon_{i'j'k'} F_{i'} F_{j'} F_{k'}$$

$$\boxed{\varepsilon_{ijk} F_i F_j F_k = \text{singlet}}$$

SU(2)

• $\Sigma = T_i \varphi_i = \frac{\sigma_i}{2} \varphi_i \quad i = 1, 2, 3$

• $S = \sigma_2 \frac{\sigma_i}{2} \Delta_i \quad (S^T = S)$

- $A = \underbrace{\sigma_2 a}_{\text{singlet}} \quad (A^T = -A)$

$$2 \times 2 = 3 + 1$$

$$2 \times \bar{2} = 3 + 1$$

$$F_2 \to U_2 F_2$$

$$\tilde{F}_2 \equiv \sigma_2 F_2^* \to U_2 \tilde{F}_2$$

$$\boxed{\text{Construct ew} = SU(3)}$$

q and l ?

Possible ?

$T_3\,,\ T_8 \equiv$ Cartan

$T_3 = \frac{1}{2}\lambda_3 \qquad T_8 = 1/2\,\lambda_8$

$$\boxed{Q = a\,T_3 + b\,T_8}$$

$Y \propto T_8$!

$$SU(3)\left\{\begin{pmatrix} 1 \\ 2 \\ \hline 3 \end{pmatrix}\right. \begin{matrix} \left.\vphantom{\begin{matrix}1\\2\end{matrix}}\right\} SU(2) \\ \vphantom{3} \end{matrix}$$

$$F = \begin{pmatrix} u \\ d \\ \hline s \end{pmatrix} \quad \text{singlet of } SU(2)$$

$u, d = u, d$ quarks

$u, d = \nu, e$ lepton

$$\left.\begin{aligned} p &= uud \\ n &= udd \end{aligned}\right\} \quad Q_p - Q_n = \\ = Q_u - Q_d = 1$$

$$Q = a T_3 + b T_8$$

$$T_8 \propto \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$$

$$\Rightarrow \quad Q_u - Q_d = a \frac{1}{2} - a\left(-\frac{1}{2}\right)$$

$$= a$$

$$\boxed{a = 1}$$

$$T_8 = \frac{1}{2}\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$$

$$\boxed{Q = T_3 + b\,T_8 \Rightarrow \mathrm{Tr}\,Q = 0}$$

$$\ell = \begin{pmatrix} u \\ d \\ \hdashline s \end{pmatrix}_L$$

$$Q_u = 1/2 + b\frac{1}{2\sqrt{3}}\,1$$

$$Q_d = -1/2 + b\frac{1}{2\sqrt{3}}\,1$$

$$Q_s = -2b\frac{1}{2\sqrt{3}}$$

$$Q_u + Q_d + Q_s = 0$$

$$2/3 - 1/3 - 1/3 = 0$$

$s = D$ – new down
Quark

$\boxed{\text{beauty of YM}}$

= non Abelian gauge group

$F \to UF$

$\rightarrow \boxed{\underline{SU(2)}}$ $\qquad \begin{pmatrix} u \\ d \end{pmatrix} \qquad \begin{pmatrix} \nu \\ e \end{pmatrix}$

$\qquad Y_2 = 1/3 \qquad Y_e = -1$

$$l = \begin{pmatrix} \nu \\ e \\ --- \\ s \end{pmatrix} \to Q_s = +1$$

No way of having

both quarks and

leptons

$\Downarrow$

GUT = only way

A (unified) model of

leptons

Georgi,
Glashow '72

SO(3) model
of leptons

$$\begin{pmatrix} \nu \\ e \\ e^c \end{pmatrix}_L$$

$$\boxed{(e^c)_L \equiv C\, e_R^T}$$

singlet

$$\boxed{3^*}$$

$$\boxed{\begin{array}{l} 3 \to U\,3 \\ 3^* \to U^*\,3^* \end{array}}$$

no connection between U and U^*

SU(2) again

$$U = e^{iH} \qquad h = \frac{\sigma_a}{2}\theta_a \qquad a = 1,2,3$$

$$\boxed{\sigma_a \neq \sigma_a^*}$$

$$U \neq U^*$$

$$\tilde{F} = \sigma_2 F^* \longrightarrow$$

$$\sigma_2 U^* F^*$$

$$= U \sigma_2 F^*$$

$$\boxed{\sigma_2 U^* = U \sigma_2}$$

$$\boxed{U^* = \sigma_2 U \sigma_2}$$

SO(2)

$$2 \times 2 = 3 + 1$$

$$2 \times \bar{2} = 3 + 1$$

$s \times$ anti s

$s = 0$

$(s \times s)$ symmetric

SO(3)

$$\boxed{\begin{aligned} 3 \times 3 &= 6 + 3^{*} \\ 3 \times 3^{*} &= 8 + 1 \end{aligned}}$$

$SU(2)$

$$\Sigma \to U \Sigma U^\dagger \Rightarrow \Sigma = \frac{\sigma_a}{2} \varphi_a$$

$$S \to U S U^T \Rightarrow S = \sigma_2 \, \sigma_a/_2 \, s_a$$

$$\boxed{\epsilon = i\sigma_2}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$