LMU GUT Course

Lecture III

10/11/2020

$$\boxed{\text{Magnetic monopoles and charge quantization}}$$

$$SU(2) \longrightarrow \boxed{Q = T_3}$$

$$[T_3, T_\pm] = \pm T_\pm$$

$$T_\pm = T_1 \pm i T_2$$

$$\boxed{T_3 = \pm n \frac{1}{2}}$$

$$\boxed{\exists \text{ monopole}}$$ Dirac

QM of electron + monopole

$$\vec{E} = \frac{q_e}{4\pi r^2}\hat{r} \longleftrightarrow \vec{B} = \frac{g_m}{4\pi r^2}\hat{r} \quad (1)$$

electric

magnetic

$$g_m = 0 \longrightarrow \vec{B} = \nabla \times \vec{A}$$

$$\nabla \cdot \vec{B} = \nabla \cdot (\nabla \times \vec{A}) = 0$$

point magnetic charge at $r = 0$

$$\rho_m \propto \delta(\vec{r})\, g_m$$

Let us try:

Yang, Wu ?

$\vec{B} = \nabla \times \vec{A} \Longrightarrow$ singularity ?

$$\vec{A}_N = \frac{g_m}{4\pi r}\,\frac{1-\cos\theta}{\sin\theta\, r}\,\hat{\varphi} \qquad (2)$$

($\theta = \pi$ singular at) world, but $\theta = \pi$ line

$$\vec{A}_S = -\frac{g_m}{4\pi r}\,\frac{1+\cos\theta}{\sin\theta\, r}\,\hat{\varphi} \qquad (3)$$

singularity at $\theta = 0$

$$\nabla \times \vec{A} = \frac{1}{r\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\, A_\varphi\right)\hat{r} + \underbrace{\cdots}_{0}$$

$$\Rightarrow \boxed{\vec{B}_N = \vec{B}_S = \frac{g_m}{4\pi r}\,\hat{r}} \qquad (4)$$

$$\vec{A}_N - \vec{A}_S = \frac{g_m}{2\pi r \sin\theta}\,\hat{\varphi} \qquad (5)$$

$$\nabla f = \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\hat{\varphi} + \cancel{\frac{\partial f}{\partial r}\hat{r}} + \cancel{\frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta}} \qquad (6)$$

$$\Downarrow$$

$$\boxed{\begin{aligned} \vec{A}_N - \vec{A}_S &= \frac{1}{e}\nabla\alpha \qquad (7) \\ \alpha &= \frac{g_m e}{2\pi}\varphi \qquad (8) \end{aligned}}$$

$$\alpha(2\pi) \neq \alpha(0)$$

charged particle $q\ (e)$

$$\psi \to e^{iq\alpha(x)}\psi \qquad (9)$$

must be single-valued

$$e^{iq\alpha(2\pi)} = e^{iq\alpha(0)}$$

$$e^{iq[\alpha(2\pi)-\alpha(0)]} = 1 \qquad (10)$$

$$\alpha = \frac{g_m e}{2\pi}\varphi$$

$$\Rightarrow \frac{q\, g_m e}{\cancel{2\pi}} \cdot \cancel{2\pi} = 2\pi n \qquad (11)$$

$$\Rightarrow \boxed{q\, g_m \cancel{e} = 2\pi n} \qquad (12)$$

$$\boxed{\text{charge is quantized !}}$$

monopole + charged particle
(e)

$\Downarrow$

$$\boxed{\begin{array}{c}\text{single-valuedness of } \psi(e) \\ \text{implies} \\ g_m\, e \propto n\end{array}}$$

$q_e = 3 q_d \qquad q_\nu = 0$

(10^{-20} precision)

QED high precision

$$\alpha_{em} = \frac{e^2}{4\pi} \simeq \frac{1}{100} \qquad (e \simeq 1/3)$$

$$\alpha_w \simeq \frac{\alpha_{em}}{\sin^2\theta_w} \simeq \frac{1}{30}$$

$$g_m e = 2\pi \qquad (n = 1)$$

$$g_m = \frac{2\pi}{e} \simeq 30$$

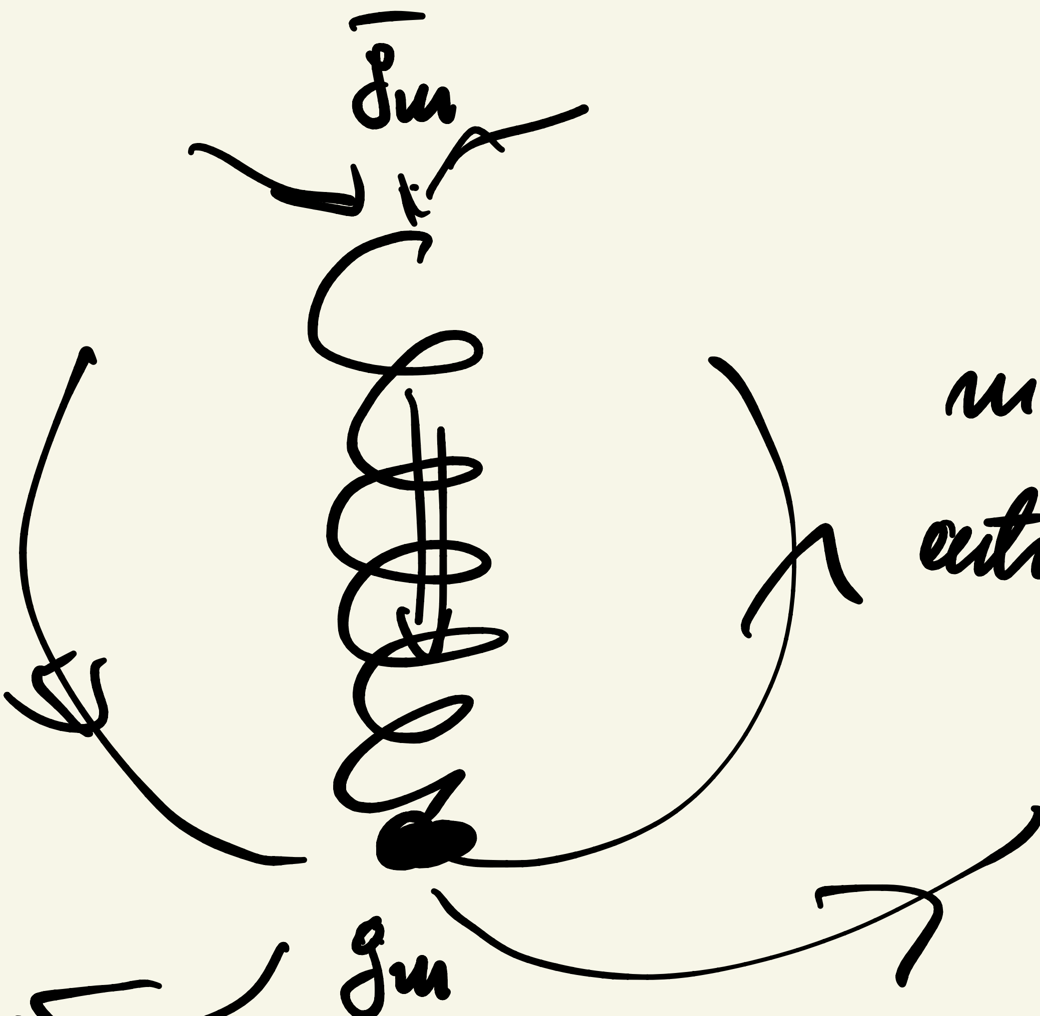
$$\boxed{\alpha_m = \frac{g_m^2}{4\pi} \simeq 100 \simeq 10^4 \alpha_{em}}$$

(strong coupling)

Dirac formulation

$$\vec{A}_S = -\frac{g_m}{4\pi r}\frac{1+\cos\theta}{\sin\theta}\,\hat{\varphi} \qquad (13)$$

$(\theta = 0)$ singular



monopole-
anti monopole

$$\boxed{\begin{array}{l}\text{Long solenoid } (\infty) \\ \text{thin} \quad \text{-"-} \quad (0) \\ \quad\quad \Updownarrow \\ \text{magnetic monopole}\end{array}}$$

$\boxed{\text{back to Wu - Yang}}$

$$\vec{A}_N - \vec{A}_S = \frac{1}{e} \nabla \alpha$$

$$\oint (\vec{A}_N - \vec{A}_S)\, d\vec{l} = \alpha(2\pi) - \alpha(0) = g m$$

$$\Big\| \qquad \left(\alpha = \frac{g m}{2\pi} \varphi \right)$$

$$\oint \vec{A}_N d\vec{l} - \oint \vec{A}_S d\vec{l}$$

$$\int \vec{B}_N\cdot d\vec{S} \overset{\parallel}{+} \int \vec{B}_c \, d\vec{S} = \int \vec{B}\, d\vec{S} = \Phi$$

magnetic flux

$$\boxed{\Phi = g_m}$$

$$\vec{B} = \frac{g_m}{4\pi r^2}\hat{r}$$

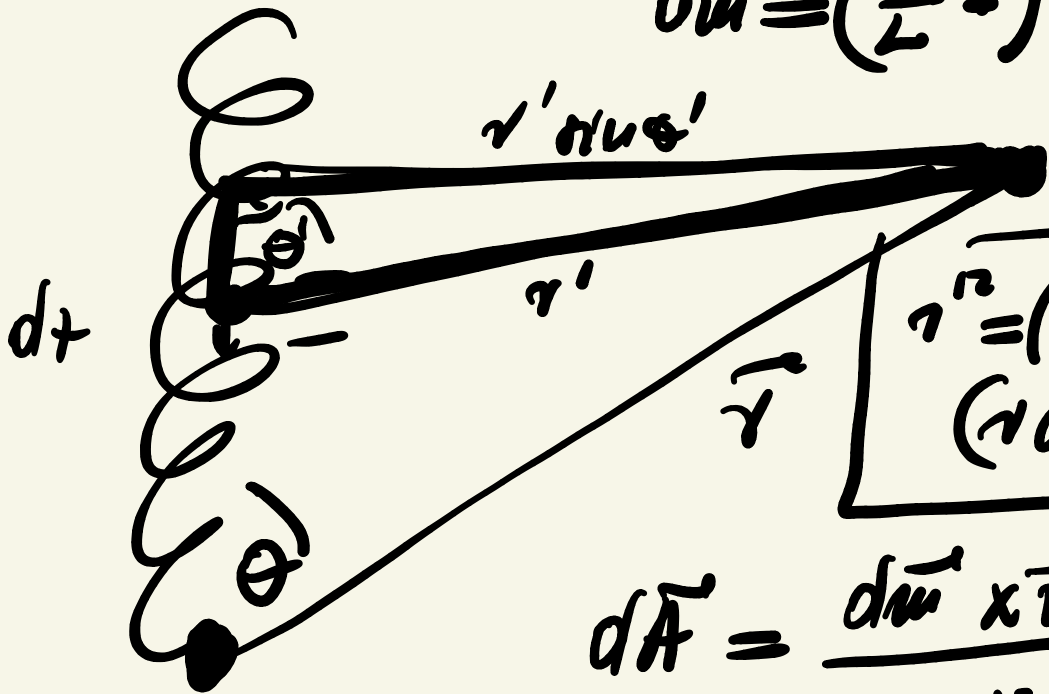
(equivalent)

$$\vec{A} = \frac{\vec{m}\times\vec{r}}{r^3}$$

potential of a dipole

$$I = \left(\frac{Q}{t}\right)$$

$$d\vec{m} = -\left(\frac{I}{L}S\right)\hat{z}\,dz$$



$$\boxed{r'^2 = (r\sin\theta)^2 + (r\cos\theta - z)^2}$$

$$d\vec{A} = \frac{d\vec{m}\times\vec{r}'}{r'^3}$$

$$\vec{A} = \underbrace{-\left(\frac{I}{L}S\right)}_{\text{dimensionless}} \int \frac{dz\,\hat{z}\times\vec{r}'}{r'^3}$$

$$\hat{z}\times\vec{r}' = r'\sin\theta'\,\hat{\varphi}$$

$$\boxed{r'\sin\theta' = r\sin\theta}$$

$$\vec{A} = -\left(\frac{IS}{L}\right) r\sin\theta \int_0^\infty \frac{dz}{r'^3}\,\hat{\varphi}$$

$$\int_0^\infty \frac{dz}{\left[(r\sin\theta)^2 + \underbrace{(z - r\cos\theta)}_{x}{}^2\right]^{3/2}}$$

$$= \int_{-r\cos\theta}^{\infty} \frac{dx}{\left[r^2\sin^2\theta + x^2\right]^{3/2}}$$

$$x = r\cos\theta\, y$$

$$= \int_{-\cot\theta}^{\infty} \frac{dy}{\left[1+y^2\right]^{3/2}} \frac{1}{(r^2\sin^2\theta)}$$

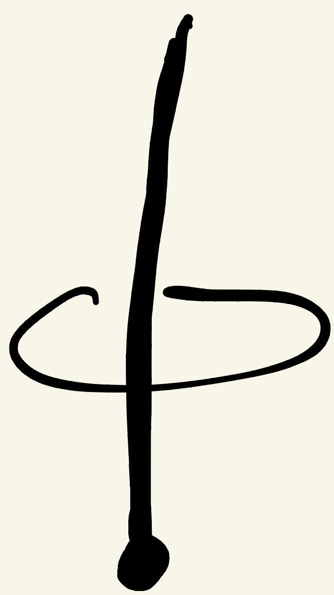
$$y = \tan u$$

$$= \frac{1}{r^2\sin^2\theta}\left[\int_{-\cos\theta}^{1} \cos u\, du = d(\sin u) = [du]\right.$$

$$\Downarrow$$

$$\vec{A}\ (\text{solenoid thin, long}) = -\left(\frac{IS}{L}\right)\frac{1}{r\sin\theta} \times (1+\cos\theta)\,\hat{\varphi}$$

$$\frac{g_m}{4\pi}$$



$$\vec{A}_N = -\frac{g_m}{4\pi r}\,\frac{1+\cos\theta}{\sin\theta}\,\hat{\varphi}$$

near ($\theta = 0$)

$$\to -\frac{g_m}{2\pi r}\,\frac{1}{\sin\theta}\,\hat{\varphi}$$

= (pure gauge)

$$= \frac{1}{e}\alpha$$

$$\alpha = \frac{g_m e}{2\pi}\varphi$$

$(\xi = 1)$

$$\boxed{\text{non-physicality of Dirac string}}$$

$$\oint \vec{A}\, d\vec{l} \neq 0 = \Delta\alpha$$

$$= \alpha(2\pi) - \alpha(0)$$

$\parallel$

flux

$$\oint A_\mu dx^\mu = \cancel{\int A_0 dt} - \oint \vec{A} d\vec{l}$$

monopole

$\boxed{\text{dyon}}$ (Q, gm)