LMU GUT Course

Lecture IV

13/11/2020

Fall 2020

Non - Abelian Monopoles

Derived:

$$\boxed{\vec{A}_s = -\frac{g}{4\pi r}\,\frac{1+\cos\theta}{\sin\theta}\,\hat{\varphi}}$$

$\theta = 0$

singular

Dirac:

(no string)

$\theta = 0$: $\quad \vec{A}_s = \frac{1}{e}\nabla\alpha \quad (r, \theta, \varphi)$

$$\alpha = \frac{g}{2\pi}\varphi$$

$$\frac{1}{e}\nabla\alpha = \frac{1}{ei}U^{\dagger}\nabla U$$

$$U = e^{i\alpha}$$

$$\boxed{U(2\pi) = U(0)}$$

$\boxed{\text{Amast!}}$

<u>SU(2)</u>

$$\psi \to e^{i\,T_a\theta_a}\psi$$

$$T_a = \sigma_a/2 \qquad (\theta_3, T_3)$$

$$\to \psi(2\pi) = e^{i\frac{\sigma_3}{2}2\pi} = -1$$

$$= e^{i\sigma_3\pi}$$

$U(1)$: $U = e^{i\alpha(x)Q}$

$U(2\pi) \neq \mathbb{1}$

$Q=1$ $U = e^{i\alpha}$

$Q = 0.1345$ $U = e^{i\alpha\, 0.1345}$

$Q = 1/3$ $U = e^{i\alpha/3}$

$SU(2)$! $Q = T_3$

$$\boxed{U = e^{iT_3\theta}}$$

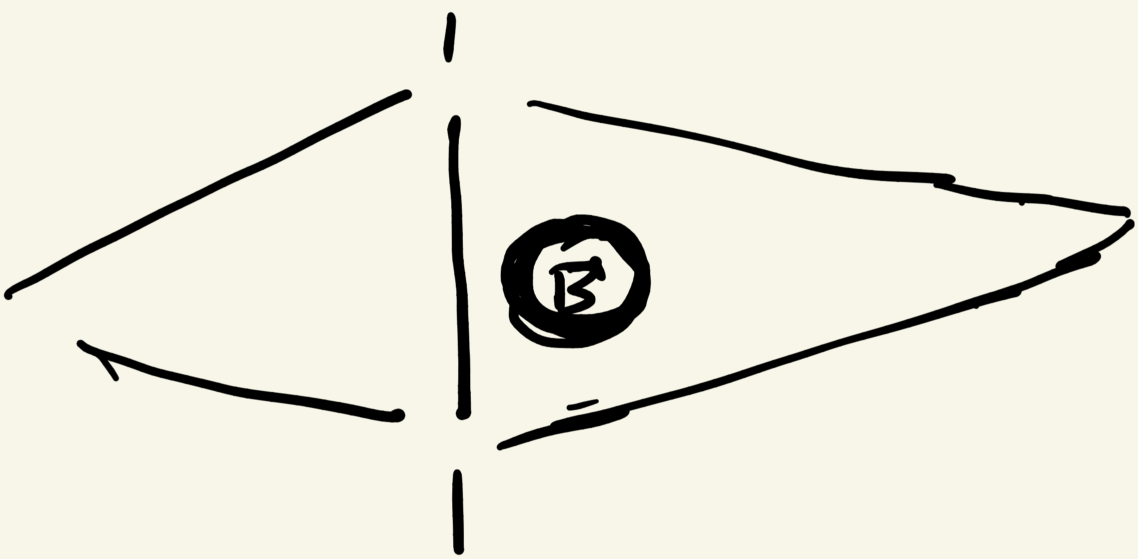
Aharonov – Bohm

$\boxed{\vec{A} = 0}$ $\psi(x) \leftrightarrow \nabla(x)$

$$\vec{A} \neq 0 \Rightarrow \psi'(x) = e^{ig(x)} \psi(x)$$

$$g(x) = e \int^{x} \vec{A} \, d\vec{x}'$$

$$\vec{\nabla} \rightarrow \vec{\nabla} - ie\vec{A}$$



$$\text{difference} = e\oint \vec{A}\, d\vec{x}$$

$$= e\int \vec{B}\, d\vec{S} = e g_m$$

$\boxed{\text{NOT physical}}$

$$\boxed{e g_m = 2\pi n} \quad (1)$$

if true $\Rightarrow$ no way of detecting "string"

$\boxed{\vec{A}_N, \ \vec{A}_S}$

Wu – Yang

$$\vec{A}_N - \vec{A}_C = \frac{1}{e} \nabla \alpha \quad (2)$$

pure gauge

iff $U = e^{i\alpha}$ (3)

$$\boxed{U(2\pi) = U(0)}$$

$$\boxed{(\vec{A}_N, \vec{A}_S) = \text{potential}}$$

$$\boxed{eg = 2\pi n}$$

Obs: $\bar{\psi}\psi$

$$\psi \to \psi' = e^{i\beta}\psi$$

$$\boxed{\oint \bar{A}\, d\bar{l} = \text{quantized}}$$

$$\oint \bar{A}\, d\bar{l} = n$$

$$\Rightarrow \boxed{\bar{A} = 0 \Leftrightarrow \bar{A} \neq 0}$$

$$\boxed{U(1) \subseteq SU(2)}$$

$$\downarrow$$

$\boxed{\text{ew}}$ (P)

$$Q = T_3$$

$$\boxed{SU(2) \longrightarrow U(1)}$$

$SO(3) \longrightarrow U(1) = SO(2)$

$\phi_i \hat{x}_i = \vec{\Phi}\,,\quad \Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$

$\Phi \rightarrow O\,\Phi \qquad O^T O = O O^T = 1$

$\det O = 1$

$\phi_i \longrightarrow \phi_i + \varepsilon_{ijk} \theta_j \phi_k$

$O = e^{i L_a \phi_a} \qquad a = 1,2,3$

$i = 1,2,3$

$L_a^\dagger = L_a,\ \mathrm{Tr}\, L_a = 0$

$(L_a)_{bc} = -i\,\varepsilon_{abc}$

<u>$SU(2)$</u> $\quad U U^\dagger = 1,\ \det U = 1$

$U = e^{i \theta_i T_i} \qquad \boxed{T_i = \frac{\sigma_i}{2}}$

$$\boxed{\Sigma \to U \Sigma U^+} \quad (*)$$

$\mathrm{Tr}\,\Sigma = 0$

$\Sigma = \Sigma^+$

$$\boxed{\Sigma = T_a \varphi_a}$$

$$\Sigma \to \hat{U}\Sigma = e^{i\theta_a \hat{T}_a}$$

$$\boxed{\hat{T}\Sigma = [T, \Sigma]} \quad (**)$$

$$V = \frac{\lambda}{4}\big(\underbrace{\varphi_i \varphi_i}_{= \vec{\varphi}^2} - v^2\big)^2 \quad (4)$$

$$\boxed{V = \frac{\lambda}{4}\left(2\mathrm{Tr}\,\Sigma^2 - v^2\right)^2 \quad (5)}$$

$$\mathrm{Tr}\,\Sigma^2 = \varphi_i \varphi_j\, \mathrm{Tr}\, T_i T_j = \varphi_i \varphi_j \tfrac{1}{2}\delta_{ij}$$

$$= \tfrac{1}{2}\vec{\varphi}^2$$

$$U = e^{i\theta_a T_a}$$

$$\Sigma \to e^{i\theta_a T_a} \Sigma e^{-i\theta_a T_a}$$

$$= (1 + i\theta_a T_a + \cdots) \Sigma (1 - i\theta_b T_b)$$

$$= \Sigma + i[T_a, \Sigma]\theta_a$$

$$= \Sigma + i\hat{T}_a \theta_a \Sigma = \hat{U}\Sigma$$

$$\hat{T}_a \Sigma = [T_a \Sigma]$$

$$\Sigma \to \Sigma + i[T_a, T_b]\varphi_b\theta_a + \cdots$$

$$= \Sigma + i\,\varepsilon_{abc} T_c \varphi_b \theta_a + \cdots$$

$$= T_c[\varphi_c + i\,\varepsilon_{abc}\theta_a\varphi_b + \cdots]$$

$$\varphi_c' = \varphi_c + i\,\varepsilon_{abc}\,\varphi_b\,\theta_a$$

checks ✓

$$\Longrightarrow$$

$$\boxed{V = \frac{\lambda}{4}\left(2\,\mathrm{Tr}\,\Sigma^2 - v^2\right)^2}$$

$$\mathcal{M}_0 = \{\text{vacuum manifold}\}$$

$$= \{\Sigma_0\,,\ V = V_{min}\}$$

$$= \{\Sigma^0\,,\ 2\,\mathrm{Tr}\,\Sigma_0^2 = v^2\}$$

$$\Downarrow$$

$$\boxed{i = 1,2,3 \qquad \varphi_0^i\,\varphi_0^i = v^2}$$

$$= S_2 \qquad \Sigma = T_a\,\varphi_a$$

vector. $\varphi_0^3 = v, \quad \varphi_0^1 = \varphi_0^2 = 0$

$$\boxed{\Sigma^0 = l_3 v}$$ $SO(3)$

$$\boxed{Q = l_3} \qquad \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}$$

$$(l_3)_{ij} = -i\varepsilon_{3ij}$$

$$l_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

photon

$$\boxed{A = A_3}$$

$$D_\mu = \partial_\mu - ig\, l_3 t_\mu^a$$

$$= \partial_\mu - ig\left(l_3 A_\mu^3 + \cdots\right)$$

$$D_\mu \Phi_0 = -ig\, l_3 A_\mu^3 \Phi_0 + \cdots$$

$$l_3 \Phi_0 = 0$$

$$\Rightarrow \boxed{A_\mu^3 = \text{massless}}$$

SU(2)

$$\hat{T}_a \Sigma = [T_a, \Sigma]$$

Pauli

$$\Sigma_0 = v T_3 \Rightarrow Q = T_3$$

$$\Rightarrow \boxed{Q = \frac{\Sigma_0}{v}}$$

***

$$[Q, \Sigma_0] = 0$$

$$G \longrightarrow H$$

Ajoint

unbroken

$\Downarrow$ $\boxed{g = e}$

$$\textcircled{e} \begin{cases} A = A_3 \\ m_A = 0 \end{cases}$$

$m_{A_1} = m_{A_2} = ev$

$\Updownarrow$

$$W^{\pm} = \frac{A_1 \mp i A_2}{\sqrt{2}}$$

$$Tr\, (D_\mu \Sigma)^+ (D^\mu \Sigma)$$

$\boxed{\text{check}}$

$$A \longrightarrow F_{\mu\nu} = \partial_\mu A_\nu^3 - \partial_\nu A_\mu^3$$

$$\neq F_{\mu\nu}^3$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\, \varepsilon_{abc} A_\mu^b A_\nu^c$$

$$F_{\mu\nu}^3 = \underbrace{(\partial_\mu A_\nu - \partial_\nu A_\mu)}_{F_{\mu\nu}} + g(A_\mu^1 A_\nu^2 - A_\mu^2 A_\nu^1)$$

$\Sigma_0 = T_a \phi_a^0$

$Q = \frac{\Sigma_0}{v}$ $\quad [Q, \Sigma_0] = 0$

unbroken

$$\boxed{SU(2) \xrightarrow[\Sigma]{} U(1)}$$

$\langle 0 \rangle$ ⟵ denote vacuum value of ...

$$\boxed{\Sigma_0 = \text{vev of } \Sigma}$$

$E_0 = 0$ (vacuum)

$V = \frac{\lambda}{4}\left(2 \mathrm{Tr}\, \Sigma^2 - v^2\right)^2$

$\mathrm{Tr}\,\Sigma_0^2 = v^2/2$

Can I find a finite energy solution (classical) of equation of motion ?

STATIC

$$\boxed{\begin{aligned} \partial_0 \Phi_{cl} &= 0 \\ D_0 \Phi_{cl} &= 0 \end{aligned}}$$

$$\mathcal{L} = \frac{1}{2}|D_\mu \Sigma|^2 - V(\Sigma) - \frac{1}{4} F_{\mu\nu}^a F^{\mu\nu}_a$$

$$\Downarrow \qquad \longrightarrow 0 \text{ at } \infty$$

$$E = \int \mathcal{H} = \int \Big[\frac{1}{2} |D_i \Phi|^2 + V(\Phi)$$

$$+ \sum_a \frac{1}{2} \left(\vec{E}_a^{\,2} + \vec{B}_a^{\,2} \right) \Big] dV$$

$$= \text{finite} \quad (< \infty)$$

$$\Rightarrow \boxed{\mathcal{H}(\Phi) \xrightarrow[\infty \, (M_\infty)]{} 0}$$

$$\Rightarrow \qquad \therefore \quad V(\Phi_\infty) = 0$$

$$\Phi_\infty \in M_0$$

$$\boxed{\text{map: } M_\infty \longrightarrow M_0}$$

$$M_0 = S_2 \Rightarrow M_\infty = S_2$$

spherically symmetry Σ_∞

$M_\infty \; (r \to \infty) \; (\theta, \varphi)$

$\Sigma_0 = v T_3$ everywhere $\longrightarrow$ at M_∞

vacuum

$$\boxed{M_\infty = S_2 \longrightarrow 1 \text{ pt} \quad \text{Trivial}}$$

$\boxed{\text{non-trivial map}}$

$\Sigma_\infty = T_a \varphi_a^\infty$ (value at ∞)

$$Q_\infty = \frac{\Sigma_\infty}{v} \quad (\text{assumed})$$

$$\hat{Q}_\infty \Sigma_\infty = [Q_\infty, \Sigma_\infty] = 0$$

Vacuum:

$$Q = v T_3 \implies A = A_3$$

$$Q = \varphi_a^0 T_a \implies A = A_a \frac{\varphi_a^0}{v} \ (?)$$

$$\sum_{a=1}^{3} \varphi_a^0 \varphi_a^0 = v^2$$

assumed

$$\left\{ \begin{array}{l} \varphi_a^0 = v\,\delta_{a3} \\ \implies A = A_3 \end{array} \right.$$

$$\left\{ \begin{array}{l} \varphi_a^0 = v\,\delta_{a1} \\ \implies A = A_1 \end{array} \right.$$

$\mathrm{Tr}\,\frac{1}{2}|D_\mu \Sigma|^2 \longrightarrow A_a A_b \underline{M}^2_{ab}$

f-n of $(\Sigma_0 = T_a \varphi_a^0)$

$$(\underline{M}^2 A^0)_a = M^2_{ab} A_b^0 = 0$$

photon

$\Updownarrow$

$$\boxed{\text{derive } \underline{M}^2_{ab}\text{, prove that } A^0 = A_a \frac{\varphi_a^0}{v} \text{ is massless}}$$

$$Q = T_a \frac{\phi_a^0}{v} \Leftrightarrow A^0 = A_a \frac{\phi_a^0}{v}$$

$$A^0 \equiv A$$

$$Q = \mathrm{Tr}\,\frac{\varphi_a^\infty}{v} \Longleftrightarrow A = \frac{A_a \varphi_a^\infty}{v}$$

$$F_{\mu\nu} = F_{\mu\nu}^a \frac{\varphi_a^\infty}{v}$$

$$\boxed{\varphi_a^\infty = v\,\frac{x^a}{r}}$$

hedgehog

$$A = A_a \frac{x_a}{r}$$

$$F_{\mu\nu} = F_{\mu\nu}^a\, x_a/r$$

$r \to 0$? $\cancel{\varphi_a = \varphi_a^\infty = v\frac{x_a}{r}}$??

φ^a_∞ $(r \to \infty)$

$\varphi^a_{\text{monopole}} \to \varphi^a_\infty$ $(r\to\infty)$

$\varphi^a_{\text{monopole}} \to 0$ $(r \to 0)$

$$\boxed{V(r=0) = \frac{\lambda}{4} v^4}$$

λ

$\Sigma_\infty(\varphi_a^\infty)$

$\phi_a = 0$ $(\Sigma = 0)$

$$E = \text{finite} \Longrightarrow D_i \Sigma_\infty = 0$$

$\Downarrow$

$$D_i \Sigma_a = \partial_i \Sigma_a - g \underbrace{\overline{\left[T_a, \Sigma_a \right] A^a_i}}\ = 0$$

$$\boxed{g = e}$$

$$\Downarrow$$

$$A^a_i(x) = x\, \cancel{\delta^a_i \frac{1}{r}} + \cancel{\gamma} \boxed{(\pm)\, \varepsilon^a_{ij} \frac{x_j}{r^2} \frac{1}{g}}$$

$$\Sigma_a = T_a \frac{x_i}{r} v + \cancel{\frac{x_i x_a}{r^2} z}$$

$$F^a_{ij}(x) = \varepsilon_{aij} \frac{1}{r^2} \frac{1}{g}$$

$$F^a_{ij} = \varepsilon_{aij} \frac{1}{gr^2} \frac{x_a}{r}$$

$$= \varepsilon_{ijk} B_k$$

$$\Rightarrow \boxed{B^a_k = \frac{1}{gr^2} \frac{x_k}{r}}$$

$$= \frac{1}{e r^2} \frac{x_k}{r}$$

$$\boxed{g_m \propto \frac{1}{e}}$$

$$g = e$$

$$\boxed{M_m \propto \hbar / g}$$