LMU GUT course

Lecture XXV

9/2/2021

$\boxed{SO(2N) \text{ spinors: III}}$

$$\boxed{SO(6)} \longleftrightarrow SO(2)$$

chiral

$$SO(10) \supseteq SO(6) \times SO(4)$$

$$G_{PS} = SU(4)_C \times SU(2)_L \times SU(2)_R$$

- $SO(4) = SU(2)_+ \times SU(2)_-$

$$\updownarrow$$

$$SO(3,1): \quad u_{L,R} \to e^{i\vec{\sigma}/2\,(\vec{\theta} \pm i\,\vec{x})}$$

$$\vec{\theta} \downarrow \text{ROT} \qquad \vec{x} \downarrow \text{BOOST}$$

$$\downarrow$$

$$SO(4) \quad U_{+,-} \to e^{i\vec{\sigma}/2\,(\vec{\theta} \pm \vec{x})}$$

$$\vec{\theta} + \vec{x} = \vec{\theta}_+$$

$$\vec{\theta} - \vec{x} = \vec{\theta}_-$$

$$\Downarrow$$

$SO(6) =$ chiral

$$SO(4): \quad \Gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \Gamma_4 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\Gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$$SO(5): \quad (\Gamma_i, \Gamma_4, \Gamma_5) \equiv \Gamma_a \quad a = 1, \dots, 5$$

$$\boxed{SO(6)} \quad \Gamma_a^{(6)} = \begin{pmatrix} 0 & \Gamma_a \\ \Gamma_a & 0 \end{pmatrix} \quad \Gamma_6 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\Gamma_{FIVE} = (-1)^3 \Gamma_1 \cdots \Gamma_6 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

- $\psi^T B \psi \qquad B = \Gamma_1 \Gamma_3 \Gamma_5$

$\qquad\qquad\qquad\quad \Gamma_2 \Gamma_4 \Gamma_6$

$\downarrow$

off-diagonal $\propto \psi_+ \psi_-$

ψ_+: our irreducible field

$\Rightarrow$ $\boxed{\underline{\text{no}} \quad \psi_+ \psi_+}$

$SO(2)$
$SO(6)$
$SO(10)$
$\downarrow$
$SO(4N+2)$

$$\Sigma_{ij} = \frac{1}{4i} [\Gamma_i, \Gamma_j]$$

$\Downarrow$

Cartan : $\{\Sigma_{12}, \Sigma_{34}, \Sigma_{56}\}$ $\boxed{r=3}$

$$\Sigma_{12} = \frac{1}{2i}\Gamma_1\Gamma_2 \Rightarrow (2\Sigma_{12})^2 = 1$$

$$\Downarrow$$

$2\Sigma_{12}$: ± 1 eigenvalues

$2\Sigma_{34}$: ± 1 – 〃 –

$2\Sigma_{56}$: ± 1 – 〃 –

$$\boxed{\psi = |\varepsilon_1\, \varepsilon_2\, \varepsilon_3\rangle}$$

$$\Gamma_{FIVE} = (-i)^3\Gamma_1 \cdots \Gamma_6 = 2\Sigma_{12}\, 2\Sigma_{34}\, 2\Sigma_{56}$$

$$= \varepsilon_1\, \varepsilon_2\, \varepsilon_3$$

- ψ_+ : $\prod_{i=1}^{3} \varepsilon_i = +1$ $\quad (\Gamma_{FIVE}\,\psi_+ = 1)$

$$\Downarrow$$

$\psi_+ :$ $\quad |+++\rangle \quad (1)$

$$\left.\begin{array}{l} |+--\rangle \\ |-+-\rangle \\ |--+\rangle \end{array}\right\} (3)$$

$$SO(6) = SU(4)$$

- $\psi_+ \longleftrightarrow$ | 4 of $SU(4)$ = fundamental

$$\Downarrow$$

$$4 = (\underbrace{3^c_{1/3} + 1^c_{-1}}_{B-L})$$

$$T_3^c = \frac{1}{2}\left(\Sigma_{12} - \Sigma_{34}\right)$$

$$T_8^c = N\left(\Sigma_{12} + \Sigma_{34} - 2\Sigma_{56}\right)$$

$$T_{15} = N' \left(\Sigma_{12} + \Sigma_{34} + \Sigma_{56} \right)$$

T_3^{ℓ}: $|+++\rangle \to 0$

2 of SU(2)

$|+--\rangle \to \frac{1}{2}$

$|-+-\rangle \to -\frac{1}{2}$ 3 of SU(3)

$|--+\rangle \to 0$

$$B-L \propto \left(\Sigma_{12} + \Sigma_{34} + \Sigma_{56} \right)$$

$$\boxed{B-L = -\tfrac{2}{3} \left(\Sigma_{12} + \Sigma_{34} + \Sigma_{56} \right)}$$

$$\Downarrow$$

$|+++\rangle$: -1 ✓

$|+--\rangle$: $+1/3$

$|-+-\rangle$: $+1/3$

$|--+\rangle$: $+1/3$ ✓

- chiral spinor of $SO(6) \longleftrightarrow$ F of $SU(4)$

$$\boxed{\psi_+ \longleftrightarrow 4_F}$$ (complex)

- opposite chirality

ψ_- : $\Gamma_{FIVE}\, \psi_- = -\psi_-$

$$\Rightarrow \prod_{i=1}^{3} G_i = -1$$

ψ_- :

$$\left.\begin{array}{l} |---\rangle \\ \\ |++-\rangle \\ |+-+\rangle \\ |--+\rangle \end{array}\right\}$$ all quantum # reversed

$\Downarrow$

$$4_- = (\bar{3}^c_{-1/3} + 1^c_{+1})$$

$$\boxed{4_- \longleftrightarrow \bar{4}}$$

spinor of SO(6) = spinor at Lorentz

$$\psi_L = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}_L$$

$$(\psi_+)_L \qquad (\psi_-)_L$$

opposite SU(4) "charges"

$$\psi_+ : \begin{pmatrix} \nu \\ e \end{pmatrix}_L \Rightarrow \psi_- : \begin{pmatrix} E^c \\ N^c \end{pmatrix}_L$$

$$(\psi_-)^c : \begin{pmatrix} N \\ E \end{pmatrix}_R$$

$\begin{pmatrix} \nu \\ e \end{pmatrix}_L$ ok

$\Downarrow$

$\bar{\nu}_L \gamma^\mu e_L W_\mu^+$

$\cancel{\begin{pmatrix} \nu_L \\ e_R \end{pmatrix}}$

$\cancel{\downarrow}$

$\cancel{\bar{\nu}_L \gamma^\mu e_R W_\mu'^+}$

$SO(6)$: $\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$

ψ_+ — irred.

ψ_- — irred.

$\{ (\psi_+)_L \quad (\psi_-)_L \}$ Green

$\{ (\psi_+)_L, \quad (\psi_-)^*_R \}$ Max

$$\begin{pmatrix} \nu \\ e \end{pmatrix}_L \qquad (e^c)_L \leftarrow \text{GUT}$$

$$\updownarrow$$

$$\begin{pmatrix} \nu \\ e \end{pmatrix}_L \qquad (e_R) \leftarrow \text{SM}$$

$\uparrow$

(charge conjugate)

$\Downarrow$

ψ_+ $\longleftrightarrow$ F (4) of $SU(4)$

$\boxed{\text{Spin}(6)}$

$\downarrow$

$$4 \times 4 = \underset{(A)}{6} + \underset{(S)}{10}$$

$SO(6) \longrightarrow$? ?

$\boxed{\text{(vector?)}}$

$$\boxed{\begin{array}{ccc} SO(3) & \xrightarrow[\text{spinor}]{} & SU(2) \\ & & \\ SO(6) & \xrightarrow[\text{spinor}]{} & Spin(6) \end{array}}$$

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6 of $SU(4)$ $\Longleftrightarrow$ vector of $SO(6)$

$$\phi_i \to O_{ij}\, \phi_j$$

- $\psi^T B \psi \propto \psi^+ \psi^-$

$$\Leftrightarrow \quad \psi_+^T B \psi_+ = 0 \quad (B = \text{off-diag})$$

$$\Rightarrow \boxed{\psi_+^T \underbrace{B \Gamma_i}_{\text{diagonal}} \psi_+ \neq 0}$$

$$\psi_+^T B \Gamma_i \psi_+ \to \psi_+^T S^T B \Gamma_i S \psi$$

$$= \psi_+^T B S^\dagger \Gamma_i S \psi \quad (\text{def. of } B)$$

$$\boxed{S^\dagger \Gamma_i S = O_{ij} \Gamma_j}$$

$$= O_{ij} \left(\psi_+^T B \Gamma_j \psi_+ \right)$$

$$\Downarrow$$

$$\boxed{\psi_+^T B \Gamma_i \psi_+ = \text{vector of } SO(6)}$$

$$\Downarrow$$

$$\mathcal{L}_Y = y \psi_+^T B \Gamma_i \psi_+ \phi_i + h.c.$$

$$\psi_+^T B \Gamma_i \psi = \underbrace{6 \text{ of } SO(6)}$$

$$\sim \phi_i$$

$$\phi_i \to O_{ij}\phi_j \iff \Phi \to O\,\Phi$$

$$O = e^{i\theta_{ij} L_{ij}}$$

$$L_{12} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & & \\ & & \cdot & - \\ & \cdot & \cdot & - \end{pmatrix} \qquad L_{ij}^T = -L_{ij}$$

$$L_{ij}^* = -L_{ij}$$

Cartan: $\{L_{12}, L_{34}, L_{56}\}$

$$(L_{ij})_{kl} = -i(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$$

$$\Downarrow$$

$$L_{12}\begin{pmatrix}1\\ i\\ 0\\ \vdots\end{pmatrix} = \begin{pmatrix}0 & -i & 0 & 0 & \cdots\\ i & \ddots & & & \\ \vdots & & & & \end{pmatrix}\begin{pmatrix}1\\ i\\ 0\\ \vdots\end{pmatrix}$$

$$= \begin{pmatrix}1\\ i\\ \vdots\end{pmatrix}$$

$$L_{12}\left(\phi_1 \pm i\phi_2\right) = \pm\left(\phi_1 \pm i\phi_2\right)$$

$$L_{34}\left(\phi_3 \pm i\phi_4\right) = \pm\left(\phi_3 \pm i\phi_4\right)$$

$$L_{56}\left(\phi_5 \pm i\phi_6\right) = \pm\left(\phi_5 \pm i\phi_6\right)$$

$$T_{3c} = \left(L_{12} - L_{34}\right)\frac{1}{2}$$

$$\boxed{6 = 3 + 3^*}$$

• $\boxed{10 \text{ of } SU(4)}$ $\quad 4 \times 4 = 6 + 10$

$$\Rightarrow \boxed{\begin{array}{l}\Delta = \overline{10} \\ \text{couples to } 4\times 4\end{array}}$$

Green: $\boxed{SO(6) = SU(4)}$

Yukawa $\longleftrightarrow$ Yukawa

• $\psi_+^T B\, \Gamma_i\, \psi_+ \sim \phi_i \quad (6)$

in $SU(4)$: $10 \subseteq 4 \times 4$

$\Leftrightarrow$ Yukawa $(F \times F)$

$\Downarrow$

$$SO(6) \qquad \underbrace{\psi_+^T \quad \psi_+}_{\text{belongs here}}$$

$$\left.\begin{array}{l} \cancel{E = ma^2} \\ \cancel{E = mb^2} \\ E = mc^2 \end{array}\right\} \text{ approach !)}$$

- $\cancel{\psi_+^T B \psi_+} = 0$
- $\psi_+^T B \Gamma_i \psi_+ \sim 6$ of $SO(6)$
- $\cancel{\psi_+^T B \underbrace{\Gamma_i \Gamma_j}_{\text{off-diag}} \psi_+} = 0 \quad (\propto \psi_+ \psi_-)$
- $\psi_+^T \underbrace{B \Gamma_i \Gamma_j \Gamma_k}_{\text{diag}} \psi_+ \neq 0$!

$\propto \quad \Phi_{[ijk]} \qquad \frac{6\cdot 5\cdot 4}{3\cdot 2\cdot 1} = 20$

$10 =$ complex repr. of $SO(4)$

$(10 = 6_c + 3_c + 1_c)$

real $\Rightarrow$ $\boxed{20 = 10 + \overline{10}}$

- $\underline{SO(2)}$ $\quad \psi_+^T B \Gamma_i \psi_+ \quad \phi_i$

$U(1)$: $\phi_1 \pm i\phi_2 \qquad (2 = 1 + \bar{1})$

$$\phi_i^{(+)} = \phi_i + \varepsilon_{ij}\phi_j$$

$$\phi_i^{(-)} = \phi_i - \varepsilon_{ij}\phi_j$$

$$\Phi^{(\pm)}_{[ijk]} = \Phi_{[ijk]} \pm \frac{i^3}{3!} \epsilon_{ijk\,i'j'k'} \bar{\Phi}_{[i'j'k']}$$

$$\boxed{20 = 10 + \overline{10}}$$

$$\boxed{(-i)^3\, \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 \Gamma_5 \Gamma_6 = +1}$$

$$SO(6) \stackrel{?}{=} SU(4)$$

$$4 \times 4 = 10 + 6$$

$$4_+ \times 4_+ = 6 + 10_{(+)}$$

Is this true?

YES !

$$\psi_+^T \underbrace{B\,\Gamma_i\Gamma_j\Gamma_k\Gamma_l}_{\text{off-diag}}\,\psi_+ = 0$$

$$(\Gamma_i\Gamma_j\Gamma_k\Gamma_l) \propto \Gamma_{i'}\Gamma_{j'}$$

- $$\psi_+^T B\,\underbrace{\Gamma_i\Gamma_j\Gamma_k\Gamma_l\Gamma_m}_{\propto\ \varepsilon_{ijklmn}\Gamma_n}\,\psi_+$$

$$(\varepsilon_{ijklmn}\,\phi_n)$$

$$\Downarrow$$

$$\boxed{G_{PS} = SO(6)\times SO(4)}$$

Anomalies

$$A_{abc} \propto \mathrm{Tr}\{T_a, T_b\} T_c$$

- $SU(2)$ = anomaly free

$$\uparrow$$

$SO(3)$ = real

$$\boxed{\text{Real repr} \quad A_{abc} = 0}$$

$$\Downarrow$$

$$\boxed{A(SO(N)) = 0} \quad \boxed{\text{Theorem?}}$$

$A(\mathrm{Spin}(N)) = 0$???

Disclaimer: careful!

$$T_a \longrightarrow \Sigma_{ij}$$

$$\mathcal{A} \propto \mathrm{Tr}\{\Sigma_{ij}, \Sigma_{ke}\}\Sigma_{mn}$$

$$\propto \underbrace{\epsilon_{ijkemn}}$$

$$\boxed{\text{exists only in } SO(6)}$$

$$\boxed{\begin{array}{c} \Rightarrow \; Spin(6) = SO(6) = SU(4) \\ \rightarrow \text{has an anomaly} \end{array}}$$

$$\Downarrow$$

$$\boxed{\begin{array}{c} \text{all } SO(2N) = Spin(2N) \\ \text{are anomaly free} \end{array}}$$

$$
\begin{aligned}
G_{PS} &= SU(4) \times SU(2) \times SU(2) \\
&= SO(6) \times SO(4) \\
&= Spin(6) \times Spin(4)
\end{aligned}
$$

$$\boxed{\text{Physics of PS}}$$

$$G_{PS} \supseteq SU(2)_L \times SU(2)_R \times U(1)_{B-L} \times SU(3)_c$$

$$\cup\!\!|$$

$$M_R \longrightarrow SU(2)_L \times U(1)_Y \times SU(3)_c$$

- $M_R \gg M_W \;(= Oh)$

$$\Rightarrow \boxed{\begin{array}{c} G_{PS} \text{ works perfectly} \\ M_{PS} \gtrsim 10^5 \text{ GeV} \end{array}}$$

$\boxed{\text{Minimal PS}}$

- minimal Higgs

(a) $(\Delta_L, \Delta_R) : (3_L, 1_R, \overline{10}_c)$
$(1_L, 3_R, \overline{10}_c)$

(b) $\overline{\Psi}_L \; \Phi \; \Psi_R$

$$\Phi(2_L, \bar{2}_R, 1_c)$$

$$\bar{2}_R = 2_R$$

1_c ← minimal

$$\Rightarrow M_f = Y_f \langle \Phi \rangle$$

$\langle \Phi \rangle = SU(4)$ singlet

$$\Psi_{L,R} = \begin{pmatrix} u & \nu \\ d & e \end{pmatrix}_{L,R}$$

$$\longleftrightarrow$$

$\nu =$ 4th color of u

$e =$ —"— of d

$$\Rightarrow \cancel{\boxed{M_d = M_e}}$$

$$M_u = M_D \quad \text{nice ?}$$

$$M_\nu = - M_D^T \frac{1}{M_N} M_D$$

$\boxed{\text{Wrong!}}$

$$\Downarrow$$

$\boxed{\text{Minimal PS fails!}}$

$\Downarrow$

$\boxed{\text{need more Higgs}}$

$\Rightarrow \quad \bar{\psi}_L \, \Phi \, \psi_R$

$$\Phi \in 4 \times \bar{4} \text{ of } SU(4)_c$$

$$= 1 + 15$$

$\Downarrow$

$\boxed{\Phi' = (2_L, 2_R, 15)}$ adjoint

$\Downarrow$

$\boxed{\text{enough?}}$

$$\langle \Phi' \rangle_{PS} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -3 \end{pmatrix} v_W$$

diagonal, traceless

in order to preserve color

$$\cancel{\boxed{M_e = -3 M_d}}$$ NOT good!

$$\boxed{M_D = -3 M_u}$$

$$\Downarrow$$

$$\boxed{\underbrace{(2,2,1) + (2,2,15)}_{\text{Minimal Higgs}}}$$

$\boxed{SO(10)}$ $\quad B = \underbrace{\Gamma_1 \Gamma_3 \Gamma_5 \Gamma_7 \Gamma_9}_{\text{off-diagonal}}$

$$\Rightarrow \Psi_+^T B \Psi_+ = 0$$

chiral ! $\quad \simeq SO(4N+2)$

- $\underbrace{\Gamma_{FIVE} = +1 \text{ on } \Psi_+}$ $\quad (r=5)$

$$\prod_{i=1}^{5} \varepsilon_i = +1$$

$$\Psi_+ = |\varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_4 \varepsilon_5\rangle$$

$$\Downarrow$$

- $|+++++\rangle \quad (1)$

- $\underbrace{|+----\rangle, |-+---\rangle, ----}_{(5)}$

• $|{+}{+}{+}{-}{-}\rangle, \ |{+}{-}{+}{-}{-}\rangle$

$$\underbrace{\binom{5}{3} = \frac{5!}{3!\,2!} = \frac{\cancel{123}45}{\cancel{123} \times 12} = 10}_{10}$$

$$\Downarrow$$

$$16_F \equiv 4_+ =$$

$$= \underbrace{1}_{\nu_R} + \underbrace{\bar{5} + 10}_{SU(5)\ (SM)}$$

$$\nu_R \Longrightarrow \boxed{m_\nu \neq 0}$$

- $SU(5)$, PS $\Rightarrow$ minimal theories ruled out!

- $SO(10)$: minimal ruled out

$\Downarrow$

$\boxed{\text{non - minimal}}$

$\boxed{SO(10)}$

$$\psi_{16} = \begin{pmatrix} u \\ d \\ \nu \\ e \\ u^c \\ d^c \\ \nu^c \\ e^c \end{pmatrix}_L$$

Even if Higgs not minimal

$\Downarrow$

$M_N \Longleftrightarrow M_Q$!

$M_D \Longleftrightarrow M_e$ (??

Minimal SM

$$l_L = \begin{pmatrix} \nu \\ e \end{pmatrix}_L \qquad e_R$$

$$\uparrow$$

$$\bar{\Phi}$$

$$\Rightarrow \mathcal{L}_Y = \bar{l}_L \, \Phi \, e_R \Rightarrow m_e \neq 0 \; \checkmark$$

$$\Rightarrow \boxed{m_\nu = 0} \text{ "fails"}$$

· add new Higgs

$$\mathcal{L}_Y' = \boxed{l_L^T C \, i\sigma_2 \, \Delta \, l_L}$$

Δ: 3 of $SU(2)_L$

$$\langle \Delta^0 \rangle \neq v_\Delta \Rightarrow m_\nu \neq 0$$

$\boxed{\text{Type II seesaw}}$

$\downarrow$

<u>prediction!</u>

$$\Delta = \begin{pmatrix} \Delta^+ & \Delta^{++} \\ \Delta^0 & -\Delta^+ \end{pmatrix}$$

$$\Delta^{--} \longrightarrow \left(\begin{array}{l} e + e \\ e + \mu \\ e + \tau \\ \mu + \tau \end{array} \right)$$

$$\boxed{Y_\Delta = \frac{M_\nu}{v_\Delta} \Leftarrow \text{know}}$$

$\Downarrow$

$$\boxed{\text{predict } e_i e_j \text{ branching ratios!}}$$

$$\boxed{M_\nu = V_e^T m_\nu V_e}$$

mixings

masses

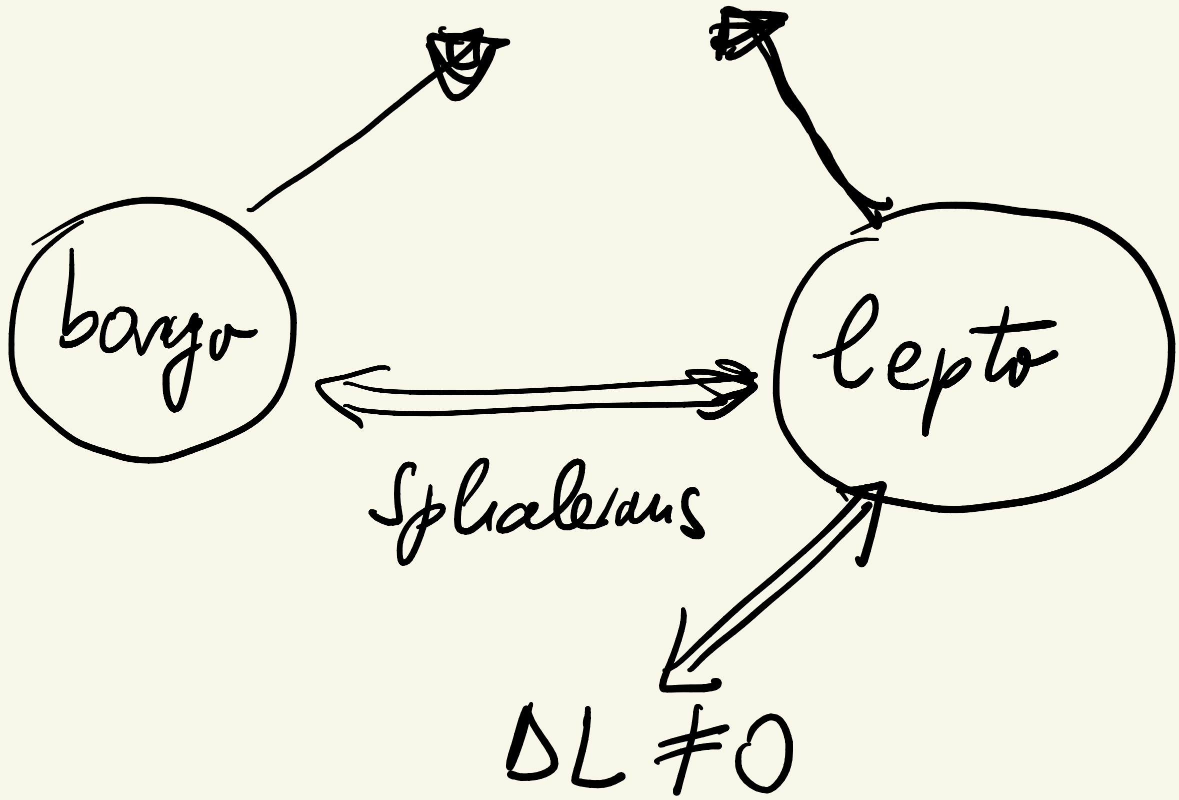
PS! ~~minimal~~ $\Rightarrow$ what next?

$\downarrow$

SO(10)

• matter asymmetry

generate dynamically
(genesis)



• $\Delta L = 2$ Majorana neutrino

$$Y_\Delta \langle \Delta \rangle = m_\nu$$

Can decays of Δ

produce asymmetry?

$\boxed{\text{NO!}}$

- $\exists\, \nu_R \Leftarrow$ type I seesaw

$\Uparrow$ (for genesis)

$\nu_R^1,\ \nu_R^2$.
