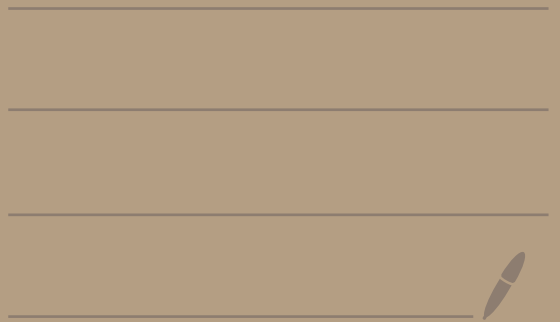


LMU GUT Course

Lecture XXIV

Feb. 5, 2021



SO(2N) spinors: II

$$\underline{SO(2)} \quad (\Gamma_1 = \Gamma_2^+)$$

$$\Gamma_i = \sigma_{i,2}$$

$$\Gamma_{\text{FIVE}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Sigma_{12} = \frac{1}{4i} [\Gamma_1, \Gamma_2] = \frac{\sigma_3}{2} \quad \parallel \quad -i \Gamma_1 \Gamma_2$$

$$\Downarrow$$
$$S = e^{i\theta_{12} \Sigma_{12}}$$

$$\Downarrow$$
$$\left[(-i)^N \Gamma_1 \cdots \Gamma_{2N} \right]$$

$$= e^{i\theta \sigma_3/2} \quad (\theta \equiv \theta_{12})$$

$$= U \Rightarrow \boxed{U+U=1}$$

$$[\Gamma_{\text{FIVE}}, \Sigma] = 0 \quad (\Gamma_{\text{FIVE}}^2 = 1)$$



$$\Gamma_{\pm} \equiv \frac{1 \pm \Gamma_{\text{FIVE}}}{2}$$

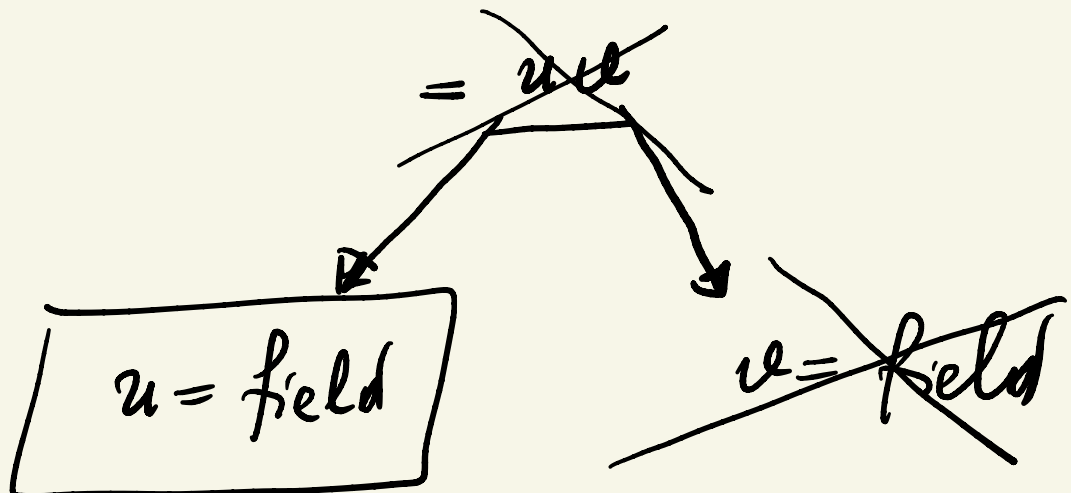
$$\Rightarrow \psi_+ = \begin{pmatrix} u \\ 0 \end{pmatrix}, \quad \psi_- = \begin{pmatrix} 0 \\ \nu \end{pmatrix}$$

$$\psi = \begin{pmatrix} u \\ \nu \end{pmatrix}$$

$$U = e^{i\theta \sigma_3 / 2} \Rightarrow \begin{array}{l} u \rightarrow e^{i\theta/2} u \\ \nu \rightarrow e^{-i\theta/2} \nu \end{array}$$

$$\bullet \psi^T B \psi \rightarrow \psi^T U^T \sigma_1 U \psi =$$

$$\stackrel{\text{///}}{=} (\sigma_1) = \psi^T \sigma_1 U^T U \psi = \psi^T \sigma_1 \psi$$



$$SO(2) = U(1) \Rightarrow \begin{array}{c} e_R \\ (u) \end{array} \Bigg| + \begin{array}{c} \cancel{e_R} \\ \cancel{(u)} \end{array}$$

Lorentz: $\underbrace{e_L \quad e_R}$

$\psi_+ = \begin{pmatrix} u \\ 0 \end{pmatrix}$ physical state

$$\psi_+^T B \psi_+ = (u \ 0) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ 0 \end{pmatrix} = 0$$

\Rightarrow no mass term

$\psi \rightarrow U\psi \Rightarrow u \rightarrow e^{i\theta/2} u$

$\underbrace{\psi^+ \psi}_{\text{mass term! ?}} = \text{inv.} \quad (\rightarrow \psi^+ \underbrace{U^+ U}_1 \psi)$

$$\psi^+ \psi = \underbrace{u^+ u}_S + v^+ v$$

$$\boxed{\psi_+^+ \psi_+ = u^+ u \checkmark} \quad \underline{\underline{\text{why not?}}}$$

$$\psi_+ \text{ --- } e_R, u_L, \dots$$

NO ! $u_L^+ u_L = \text{NOT Lorentz}$
inv!

$$u_L^T C u_L = \text{Lorentz invariant}$$

\Rightarrow $\boxed{\psi_+^T B \psi_+ \text{ is the only allowed bi-linear}}$

$\boxed{SO(2) = \text{chiral group}}$

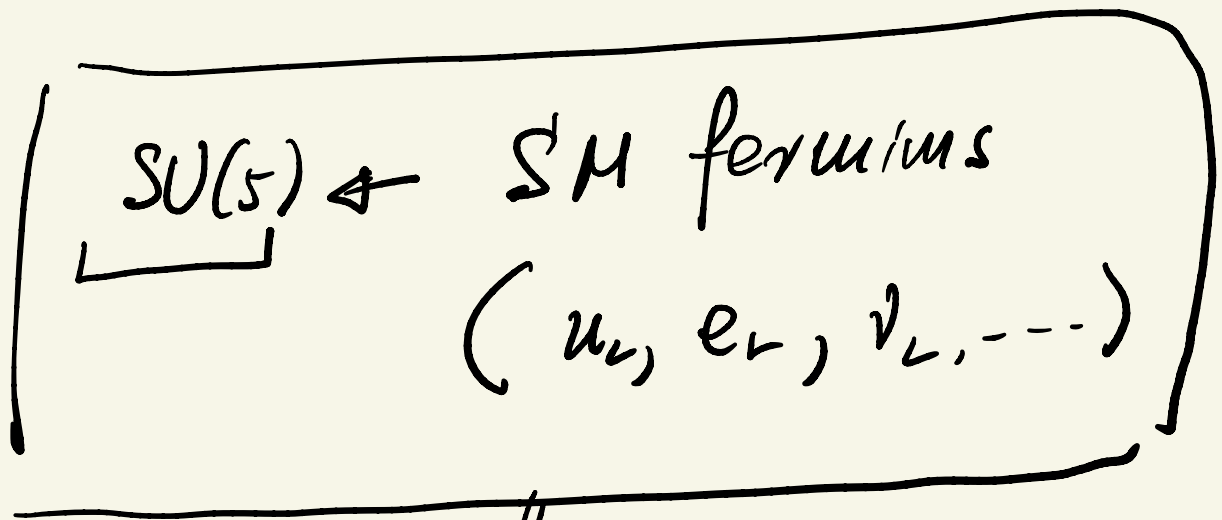
$$\underline{SO(10)} \cong SU(5)$$

• $\psi_+^\dagger \psi_+ = SO(2)$ inv. ~~$(\psi_{L,+}^\dagger \psi_{L,+})$~~

NOT Lorentz inv.

• $\psi_{+,L}^\dagger B C \psi_{+,L} = \text{Lorentz inv.}$

$= 0$ in $SO(2)$



$SO(2)$ repr.

Lorentz + internal

• Yukawa int.?

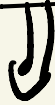
$$\phi_i \quad (i=1,2)$$

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \rightarrow O \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

$$O = e^{i\theta_{12} L_{12}} O$$

$$L_{12} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\left| (L_{ij})_{\mu\nu} = -i (\delta_{i\mu} \delta_{j\nu} - \delta_{i\nu} \delta_{j\mu}) \right|$$



$$O = e^{i\theta \sigma_2} = \cos \theta + i \sin \theta \sigma_2$$

$(\sigma_2^2 = 1)$

$$= \begin{pmatrix} \cos \theta & i \sin \theta \\ -i \sin \theta & \cos \theta \end{pmatrix}$$

$$\phi_i \rightarrow O_{ij} \phi_j \quad (2 = 1 + 1)$$

$$\psi_+^T B \Gamma_i \psi_+ \rightarrow \psi_+^T S^T B \Gamma_i S \psi_+$$

$$= \psi_+^T B \underbrace{S^T \Gamma_i S}_{\parallel}$$

$$O_{ij} \Gamma_j$$

$$= O_{ij} \underbrace{(\psi_+^T B \Gamma_i \psi_+)}_{\text{vector repn. of } SO(2)}$$

vector repn. of $SO(2)$

$$\Rightarrow \mathcal{L}_y = \psi_+^T B \Gamma_i \psi_+ \phi_i =$$

$$= (u \ 0) \sigma_1 \sigma_i \begin{pmatrix} u \\ 0 \end{pmatrix} \phi_i$$

$$= uu \phi_1 + (u \ 0) i \sigma_3 \begin{pmatrix} u \\ 0 \end{pmatrix} \phi_2$$

$$= u u \phi_1 + i u u \phi_2 = u u (\phi_1 + i \phi_2)$$

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \rightarrow \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

$$c \equiv \cos \theta, \quad s \equiv \sin \theta$$

$$\phi_1 \pm i \phi_2 \rightarrow e^{\mp i \theta} (\phi_1 \pm i \phi_2)$$

$$\bullet \quad u \rightarrow e^{i \theta / 2} u, \quad \phi_1 + i \phi_2 \rightarrow e^{-i \theta} (\phi_1 + i \phi_2)$$

$$\bullet \quad \overline{u u} (\phi_1 - i \phi_2) = \text{invariant}$$

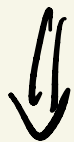
$$\psi_-^T B \Gamma: \psi_- \phi_1' = \overline{u u} (\phi_1 - i \phi_2)$$

$$\phi_{\pm} = \text{self (anti) dual}$$

$$\left\{ \begin{array}{l} \phi_i^{(+)} = (\phi_i + i \sum_{j'} \phi_{j'}) \quad \left(\frac{1}{\sqrt{2}}\right) \\ \phi_i^{(-)} = (\phi_i - i \sum_{j'} \phi_{j'}) \quad \left(\frac{1}{\sqrt{2}}\right) \end{array} \right\}$$

$$\bullet \boxed{\phi_1^{(+)} = \phi_1 + i \phi_2}$$

$$\phi_2^+ = \phi_2 - i \phi_1 = -i (\phi_1 + i \phi_2) \quad \text{not new}$$



SO(2N)

$$\underbrace{\phi_{[i j k \dots]}^{(\pm)}}_N = \underbrace{\phi_{[i j \dots]}}_N \pm \frac{i^N}{N!} \times$$

$$\times \underbrace{\sum_{j' k' \dots}}_N \underbrace{\phi_{[i' j' k' \dots]}}_N$$

self dual (+) } gen. at
 anti -1- (-) } complex
 $(\phi_1 \pm i\phi_2)$

$SO(z)$
 (Euclidean)



$SO(1,1)$
 (Minkowski)

$$\Gamma_i = (\sigma_1, \sigma_2)$$

$$\Gamma_2^+ = \Gamma_i$$



$$\boxed{(\Sigma_{ij})^+ = (\Sigma_{ij})}$$

(ROT)

$$\Gamma_{\text{FIVE}} = -i\sigma_1\sigma_2$$

$$\Gamma_0 = \sigma_1, \Gamma_2 = i\sigma_2$$

$$\Gamma_0^2 = 1, \Gamma_1^2 = -1$$



$$\Sigma_{12} = \frac{i\sigma_3}{2}$$

$$(\Sigma_{12})^+ = -\Sigma_{12}$$

BOOST

$$\Gamma_{\text{FIVE}} = \sigma_3 = -\Gamma_1\Gamma_2$$

$$\psi^T B \psi$$

$$\equiv \sigma_1$$

$$= \cancel{\psi_+ \psi_-}$$

(the same)

$$\psi^T B \psi$$

$$\equiv \sigma_1$$

$$= \cancel{\psi_+ \psi_-}$$

$SO(2N)$

$SO(3)$ (en passant)

$$\Gamma_i = \sigma_i, \quad (i=1,2,3)$$

$$SO(2) + (\Gamma_{\text{FIVE}} = \sigma_3)$$

$$\{\Gamma_{\text{FIVE}}, \Gamma_i\} = 0$$



$$\Gamma_{\text{FIVE}} \neq \Gamma_1 \Gamma_2 \Gamma_3 \neq 1$$

\Rightarrow no duality

$$\Sigma_{ij} \equiv \frac{1}{2\epsilon_{ij}} [\Gamma_{ij}, \Gamma_j] = \frac{1}{4\epsilon_{ij}} [\sigma_i, \sigma_j]$$

$$= \frac{1}{2\epsilon_{ij}} \epsilon_{ijk} \sigma_k$$

$$\Theta_{ij} = \epsilon_{ijk} \Theta_k$$

$$\Theta_{ij} \Sigma_{ij} = \frac{1}{2} \Theta_k \sigma_k$$

$$S'(so(3)) = \left| U_{2 \times 2} = e^{i \vec{\theta} \cdot \vec{\sigma} / 2} \right.$$

$$\left[\psi^T B \psi = \text{allowed?} \right]$$

$$B = \cancel{\sigma_1} \sigma_2 \quad \text{in } so(2)$$

why?

$$\psi^T B \psi \rightarrow \psi^T U^T B U \psi =$$

$$= \psi^T B U^T U \psi = \psi^T B \psi$$

$$U^T B = B U^T$$

$$\Sigma^T B + B \Sigma = 0$$

$$\sigma_2^T B + B \sigma_2 = 0$$

$$B = \sigma_2$$

$$\psi = \begin{pmatrix} u \\ d \end{pmatrix} \Rightarrow \psi^T \sigma_2 \psi =$$

$$= u d - d u$$

Lorentz

$$= \uparrow d - d \uparrow$$

$$\bullet \quad \psi_L^T i\sigma_2 C \psi_L \quad \psi_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$$

$$= - \psi_L^T C^T (i\sigma_2)^T \psi_L$$

$$= - \psi_L^T (-c) (-i\sigma_2) \psi_L =$$

$$= - \psi_L^T C i\sigma_2 \psi_L = 0$$

⇓

no mass term! ?

↳ $SU(2)$ chiral?

NO!

$$\psi_{1L}^T C i\sigma_2 \psi_{2L} = - \psi_{2L}^T C i\sigma_2 \psi_{1L}$$

$\neq 0$

- with 1 gen. \Rightarrow $SU(2) = \text{chiral}$
- more gen. \Rightarrow ~~NOT~~ NOT chiral
 \Rightarrow direct mass

different from $SU(2)$

$$\psi_1^T B \psi_2 = \psi_{1+} \psi_{2-} = u_1 d_2 \dots$$

$$\Rightarrow \psi_{1+}^T B \psi_{2+} = 0$$

~~NOT~~ Truly chiral

SO(4)

$$\Gamma_i^{(4)} = \begin{pmatrix} 0 & \Gamma_i^{(3)} \\ \Gamma_i^{(3)} & 0 \end{pmatrix}, \quad \Gamma_4 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad -11-$$

Lorentz \downarrow

$$\left[\begin{array}{l} \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \Gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array} \right]$$

$$\Gamma_{\text{FIVE}} = (-i)^N \underbrace{\Gamma_1 \dots \Gamma_{2N}}_{SO(2N)}$$

$$= (-1) \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4$$

$$= (-1) \begin{pmatrix} \sigma_1 \sigma_2 & 0 \\ 0 & \sigma_1 \sigma_2 \end{pmatrix} \begin{pmatrix} i\sigma_3 & 0 \\ 0 & -i\sigma_3 \end{pmatrix}$$

$$= \begin{pmatrix} +\mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}$$



$$\boxed{\Gamma_{\pm} = \frac{1 \pm \Gamma_{FIVE}}{2}}$$

$$\begin{aligned} \Rightarrow \psi_+ &= \Gamma_+ \psi \\ \psi_- &= \Gamma_- \psi \end{aligned}$$



$$\psi^T B \psi = i\nu\nu_0$$

$$SO(2) \Rightarrow B = \sigma_1 \quad (\sigma_2)$$



$$B = \Gamma_1 \Gamma_3 \Gamma_5 \dots$$

$$\text{or } B = \Gamma_2 \Gamma_4 \Gamma_6 \dots$$

$$B = -\sigma_1 \sigma_3 = \begin{pmatrix} +i\sigma_2 & 0 \\ 0 & +i\sigma_2 \end{pmatrix}$$

$$B = \Gamma_2 \Gamma_4 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}$$

$B = \text{diag}$

$$\Gamma_{\text{FIVE}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\psi_+ = \begin{pmatrix} u \\ 0 \end{pmatrix} \therefore \psi_+^T B \psi_+ = \underline{\underline{u^T i\sigma_2 u}}$$

$\Rightarrow \boxed{SO(4) = \text{vector-like}}$

$$\boxed{SO(4) = SU(2) \times SU(2)}$$

$$\Gamma_{\text{FIVE}} = (-i)^N T_1 \dots T_{2N}$$

$$\Sigma_{ij} = \frac{1}{2i} [T_i, T_j]$$

$$\Sigma_{12} = \frac{1}{2i} T_1 T_2, \quad \Sigma_{34} = \frac{1}{2i} T_3 T_4, \dots$$

$$\Sigma_{2N-1, 2N} = \frac{1}{2i} T_{2N-1} T_{2N}$$

$$\Gamma_{\text{FIVE}} = (2\Sigma_{12}) (2\Sigma_{34}) \dots (2\Sigma_{2N-1, 2N})$$

$$\left\{ \Sigma_{12}, \Sigma_{34}, \dots, \Sigma_{2N-1, 2N} \right\} =$$

= Cartan sub-algebra

$$\boxed{\nu(SO(2N)) = N}$$

$$2\Sigma_{12} = \frac{1}{i} \Gamma_1 \Gamma_2 \Rightarrow (2\Sigma_{12})^2 = (-1) \Gamma_1 \Gamma_2 \Gamma_1 \Gamma_2 = +1$$

$$(2 \Sigma_{2n-1, 2n}) : \pm 1 \text{ eigenvalues} \\ = e_i (\pm 1)$$

$$\psi_+ = \frac{1 + \Gamma_{\text{FIVE}}}{2} \psi = \Gamma_+ \psi$$

$$\Gamma_+ \psi_+ = \psi_+ \Leftrightarrow \\ \Gamma_+ = 1 \text{ in } \psi_+ \text{ space}$$

$$\Rightarrow \Gamma_{\text{FIVE}} \psi_+ = \psi_+$$

$$\parallel \\ \mathbb{1} \text{ in the sub-space}$$

$$\Gamma_{\text{FIVE}} = 2\Sigma_{12} \quad \dots \quad 2\Sigma_{2N-1, 2N}$$

$$\psi_+ = |\varepsilon_1 \dots \varepsilon_N\rangle \quad \therefore \quad \left(\prod_{i=1}^N \varepsilon_i = 1 \right)$$

SO(2) $\psi_+ = |++\rangle = |1+\rangle = |0+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

SO(4) $\psi_+ = |\varepsilon_1 \varepsilon_2\rangle \quad \therefore \quad \varepsilon_1 \varepsilon_2 = +1$



$$\psi_+ : \begin{array}{l} |++\rangle \\ |--\rangle \end{array}$$

$\{\Sigma_{12}, \Sigma_{34}\}$
" "
Cartan

$$(T_3)_+ = \frac{1}{2} (\Sigma_{12} + \Sigma_{34}) \quad \left[\begin{array}{l} (T_{3L}) \\ \vdots \end{array} \right]$$

$$(T_3)_- = \frac{1}{2} (\Sigma_{12} - \Sigma_{34}) \quad \left[\begin{array}{l} \vdots \\ (T_{3R}) \end{array} \right]$$

$$\left. \begin{aligned} (T_3)_+ |++\rangle &= \frac{1}{2} |++\rangle \\ (T_3)_+ |--\rangle &= -\frac{1}{2} |--\rangle \end{aligned} \right\} SU(2)_+ \#$$

$$\left. \begin{aligned} (T_3)_- |++\rangle &= 0 \\ (T_3)_- |--\rangle &= 0 \end{aligned} \right\} SU(2)_-$$

• $\psi_+ = |\varepsilon_1 \varepsilon_2\rangle \therefore \varepsilon_1 \varepsilon_2 = +1$

= doublet of $SU(2)_+$
singlet of $SU(2)_-$

• $\psi_- = |\varepsilon_1 \varepsilon_2\rangle \therefore \varepsilon_1 \varepsilon_2 = -1$

$\hookrightarrow \begin{pmatrix} |+-\rangle \\ |-+\rangle \end{pmatrix}$ singlet of $SU(2)_+$
doublet of $SU(2)_-$

$$\psi^T B \psi = \psi_+^T i \sigma_2 \psi_+ \dots$$

$SO(4) \neq$ chiral (mass term)
(vector-like)

$$SO(4) \leftrightarrow \text{Lorentz}$$

$$\psi_{1+,L}^T B C \psi_{2+,L} =$$

$$= u_{1L}^T C i \sigma_2 u_{2L} =$$

$$= -u_{2L}^T C^T (i \sigma_2)^T u_{1L} = \underbrace{-u_{2L}^T C i \sigma_1 u_{1L}}$$

$$\Rightarrow \boxed{u_{1L}^T C i \sigma_2 u_{1L} = 0} \quad \left(\begin{array}{l} \text{no mass} \\ \text{for 1 gen.} \end{array} \right)$$

⇓ more gen.

$SO(4) = \text{bad group}$
 $= \text{direct sum}$

$SO(2)$: $B = \sigma_1$ (off diagonal)

$SO(4)$ $B = \begin{pmatrix} \pm i\sigma_2 & 0 \\ 0 & \pm i\sigma_2 \end{pmatrix}$ (diagonal)

⇓

$SO(2)$ $\psi_+ \psi_+ = 0$
 $SO(4)$ $\psi_+ \psi_+ \neq 0$

⇓

$SO(4) = SU(2) \times SU(2)$

$$\underline{SO(5)} \left[\begin{array}{l} \Gamma_i^{(4)}, \Gamma_{\text{FIVE}} = \Gamma_a^{(5)} \\ i=1, \dots, 4 \qquad a=1, \dots, 5 \end{array} \right]$$

$$\boxed{SO(6)} \quad (SO(6) = SU(4))$$

$$\mathcal{G}(PS) \Downarrow = SO(4) \times SO(6)$$

$$\Gamma_i^{(6)} = \begin{pmatrix} 0 & \Gamma_i^{(5)} \\ \Gamma_i^{(5)} & 0 \end{pmatrix}, \quad \Gamma_6^{(6)} = \begin{pmatrix} 0 & -i\mathbb{1}_4 \\ i\mathbb{1}_4 & 0 \end{pmatrix}$$

$i=1, \dots, 5$

$$\Gamma_{\text{FIVE}} = (-1)^3 \Gamma_{1, \dots, 5} \quad \Gamma_6 = \begin{pmatrix} +\mathbb{1}_4 & 0 \\ 0 & -\mathbb{1}_4 \end{pmatrix}$$



$$\psi^T B \psi = m,$$

$$B = T_1 T_3 T_5$$

$$(T_2 T_4 T_6)$$

$$T_1 (\sigma_1)$$

$$T_2 (\sigma_3)$$

$$T_5 (\mathbb{1})$$

$$\Rightarrow T_1 T_3 T_5 = \begin{pmatrix} 0 & f(\sigma_2) \\ f(\sigma_2) & 0 \end{pmatrix}$$

off-diagonal

$$\Rightarrow \boxed{\psi_+^T B \psi_+ = 0}$$

\Downarrow

$$\boxed{SO(6) \sim SO(2)}$$

dual

\Downarrow

~~$SO(8) \Rightarrow R = \text{diquark (4 off diag)}$~~

$SO(10) \Rightarrow R = \text{off-diquark (5-1)}$

good

$SO(2) : 2^1 = 2^{2/2} = 2^{N/2}$

$2 = 1_+ + 1_-$

$SO(4)$: $4 = 2^2 = 2^{N/2}$

$4 = 2_{(+) } + 2_{(-)}$

$SO(6)$ $\left\{ \begin{array}{l} 15 = 6 \cdot \frac{5}{2} \\ \mathcal{F} = 2^3 = 2^{N/2} \end{array} \right.$

$r=3$



$\mathcal{F} = \left(\begin{array}{c} 4_{(+)} \\ \uparrow \\ \uparrow \end{array} \right) + 4_{(-)}$

$$\boxed{SU(4)} \quad (\nu=3)$$

Irreducible

$$(15 \text{ gen} = 4^2 - 1)$$

$$d(4) = 2^N \text{ in } SO(2N)$$

$$SO(1, 2N-1)$$