LMU GUT Course

Lecture XXIV

Feb. 5, 2021

SO(2N) spinors: II

SO(2)

$(\Gamma_1 = \Gamma_2^{+})$

$\Gamma_i = \sigma_{1,2}$ $\qquad \Gamma_{FIVE} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\Sigma_{12} = \frac{1}{4i}[\Gamma_1, \Gamma_2] = \frac{\sigma_3}{2} \qquad \Gamma_{FIVE} = -i\,\Gamma_1 \Gamma_2$

$\Downarrow \qquad\qquad \downarrow$

$S = e^{i\theta_{12}\Sigma_{12}} \qquad \left[(-i)^N \Gamma_1 \cdots \Gamma_{2N} \right]$

$= e^{i\theta \sigma_3/2} \qquad (\theta \equiv \theta_{12})$

$= U \Rightarrow \boxed{U^{+}U = 1}$

$[\Gamma_{FIVE}, \Sigma] = 0 \quad (\Gamma_{FIVE}^2 = 1)$

$\Downarrow$

$$\Gamma_\pm \equiv \frac{1 \pm \Gamma_{FIVE}}{2}$$

$$\Rightarrow \quad \psi_+ = \begin{pmatrix} u \\ 0 \end{pmatrix}, \quad \psi_- = \begin{pmatrix} 0 \\ v \end{pmatrix}$$

$$\psi = \begin{pmatrix} u \\ v \end{pmatrix}$$

$$U = e^{i\theta \sigma_3/2} \Rightarrow \boxed{\begin{array}{l} u \to e^{i\theta/2} u \\ v \to e^{-i\theta/2} v \end{array}}$$

- $\psi^T B \psi \to \psi^T U^T \sigma_1 U \psi =$

$$B \equiv (\sigma_1) \qquad = \psi^T \sigma_1 U^+ U \psi = \psi^T \sigma_1 \psi$$

$$= \cancel{uv}$$

$$\boxed{u = \text{field}} \qquad v = \cancel{\text{field}}$$

$$SO(2) = U(1) \Rightarrow \begin{array}{c} e_R \\ (\nu) \end{array} \Bigg| + \cancel{e_R'} \ \cancel{(\nu')}$$

$$\text{Lorentz}: \quad \underbrace{e_L \qquad\qquad e_R}$$

- $\psi_+ = \begin{pmatrix} u \\ 0 \end{pmatrix}$ physical state

$$\psi_+^T B \psi_+ = (u\ 0)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ 0 \end{pmatrix} = 0$$

$$\Rightarrow \boxed{\text{no mass term}}$$

- $\psi \to U\psi \Rightarrow u \to e^{i\theta/2} u$

$$\underbrace{\psi^+\psi}_{\text{mass term !?}} = \text{inv.} \quad (\to \psi^+ \underbrace{U^+ U}_{1} \psi)$$

$$\psi^\dagger \psi = \underbrace{u^\dagger u} + v^\dagger v$$

$$\boxed{\psi_+^\dagger \psi_+ = u^\dagger u \checkmark}$$ why not?

ψ_+ — e_R, u_L, - - - -

NO ! $u_L^\dagger u_L$ = NOT Lorentz inv !

$u_L^T G u_L$ = Lorentz invariant

$\Rightarrow$ $\boxed{\psi_+^T B \psi_+ \text{ is the only allowed bi-linear}}$

$\boxed{SO(2) = \text{chiral group}}$

$$\underbrace{SO(10)}\supseteq SU(5)$$

- $\psi_+^\dagger \psi_+ = SO(2)$ inv. $\left(\cancel{\psi^\dagger_{L,+}\psi_{L,+}}\right)$

NOT Lorentz inv.

- $\psi^T_{+,L} B C \psi_{+,L} =$ Lorentz inv.

$= 0$ in $SO(2)$

$$\boxed{\underbrace{SU(5)} \leftarrow \text{SM fermions } (u_L, e_R, \nu_L, \dots)}$$

$\Downarrow$

$SO(2)$ repr.

$$\boxed{\text{Lorentz + internal}}$$

- Yukawa int.?

$$\phi_i \quad (i=1,2)$$

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \to O \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$$

$$O = e^{i\theta_{12} L_{12}} \, O$$

$$L_{12} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\boxed{\left((L_{ij})_{kl} = -i\left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right)\right)}$$

$$\Downarrow$$

$$O = e^{i\theta\sigma_2} = \cos\theta + i\sin\theta\,\sigma_2$$

$$(\sigma_2{}^2 = 1)$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\phi_i \to O_{ij}\phi_j \quad (2 = 1+1)$$

$$\psi_+^T B \Gamma_i \psi_+ \to \psi_+^T S^T B \Gamma_i S \psi_+$$

$$= \psi_+^T B \underbrace{S^+ \Gamma_i S}_{\substack{\| \\ O_{ij}\Gamma_j}} \psi_+$$

$$= O_{ij} \underbrace{(\psi_+^T B \Gamma_i \psi_+)}_{\text{vector repr. of } SO(2)}$$

$$\Rightarrow \mathcal{L}_Y = \psi_+^T B \Gamma_i \psi_+ \phi_i =$$

$$= (u\ 0)\,\sigma_1 \sigma_i \begin{pmatrix} u \\ 0 \end{pmatrix} \phi_i$$

$$= uu\,\phi_1 + (u\ 0)\, i\sigma_3 \begin{pmatrix} u \\ 0 \end{pmatrix} \phi_2$$

$$= uu\,\phi_1 + i\,uu\,\phi_2 = uu\,(\phi_1 + i\phi_2)$$

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \to \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

$c \equiv \cos\theta, \; s \equiv \sin\theta$

$$\boxed{\phi_1 \pm i\phi_2 \to e^{\mp i\theta}(\phi_1 \pm i\phi_2)}$$

- $u \to e^{i\theta/2} u, \quad \phi_1 + i\phi_2 \to e^{-i\theta}(\phi_1 + i\phi_2)$

- $vv(\phi_1 - i\phi_2) =$ invariant

$$\psi_-^T B \Gamma_i \psi_- \quad \phi_i' = vv(\phi_1 - i\phi_2)$$

$$\boxed{\phi_\pm = \text{self (anti) dual}}$$

$$\begin{cases} \phi_i^{(+)} = \left(\phi_i + i\,\varepsilon_{ij}\,\phi_j\right)\left(\frac{1}{\sqrt{2}}\right) \\ \phi_i^{(-)} = \left(\phi_i - i\,\varepsilon_{ij}\,\phi_j\right)\left(\frac{1}{\sqrt{2}}\right) \end{cases}$$

- $\boxed{\phi_1^{(+)} = \phi_1 + i\phi_2}$

$$\phi_2^{+} = \phi_2 - i\phi_1 = -i\left(\phi_1 + i\phi_2\right)$$ - not new

$$\Downarrow$$

SO(2N)

$$\phi^{(\pm)}_{\underbrace{[ijk\ldots]}_{N}} = \phi_{\underbrace{[ij\ldots]}_{N}} \pm \frac{i^N}{N!} \times$$

$$\times\ \varepsilon_{\underbrace{ijk\ldots}_{N}\ldots\underbrace{i'j'k'}_{N}}\ \phi_{[i'j'k'\ldots]}$$

self dual (+)
anti —"— (−) $\Big\}$ gen. at complex $(\phi_1 \pm i\phi_2)$

$SO(2) \longleftrightarrow SO(1,1)$

(Euclidean) (Minkowski)

$\Gamma_i = (\sigma_1, \sigma_2)$

$\Gamma_2^+ = \Gamma_2$

$\Downarrow$

$(\Sigma_{ij})^+ = (\Sigma_{ij})$

(ROT)

$\Gamma_{FIVE} = -i\sigma_1\sigma_2$

$\Gamma_0 = \sigma_1, \quad \Gamma_1 = i\sigma_2$

$\Gamma_0^2 = 1, \quad \Gamma_1^2 = -1$

$\Downarrow$

$\Sigma_{12} = \frac{i\sigma_3}{2}$

$(\Sigma_{12})^+ = -\Sigma_{12}$

$\boxed{\text{BOOST}}$

$\Gamma_{FIVE} = \sigma_3 = -\Gamma_1\Gamma_2$

$$\psi^T B \psi \qquad \text{(the same)} \qquad \psi^T B \psi$$

$$\overset{\shortparallel}{\sigma_1} \qquad\qquad \overset{\shortparallel}{\sigma_1}$$

$$= \cancel{\psi_+ \psi_-} \qquad\qquad = \cancel{\psi_+ \psi_-}$$

$$\boxed{SO(2N)}$$

$\underline{SO(3)}$ (en passant)

$$\Gamma_i = \sigma_i, \quad (i = 1, 2, 3)$$

$$\left.\begin{array}{l} SO(2) + (\Gamma_{FIVE} = \sigma_3) \\ \{\Gamma_{FIVE}, \Gamma_i\} = 0 \end{array}\right\}$$

$$\Gamma_{FIVE} \propto \Gamma_1 \Gamma_2 \Gamma_3 \propto 1$$

$$\Rightarrow \boxed{\underline{\text{no}} \text{ chirality}}$$

$$\Sigma_{ij} \equiv \frac{1}{4i}\left[\Gamma_i, \Gamma_j\right] = \frac{1}{4i}\left[\sigma_i, \sigma_j\right]$$

$$= \frac{1}{2i}\varepsilon_{ijk}\sigma_k$$

$$\theta_{ij} = \varepsilon_{ijk}\theta_k$$

$$\theta_{ij}\Sigma_{ij} = \frac{1}{2}\theta_k\sigma_k$$

$$S(SO(3)) = \boxed{U_{2\times 2} = e^{i\vec{\theta}\cdot\vec{\sigma}/2}}$$

$$\Downarrow$$

$$\boxed{\psi^T B \psi = \text{allowed ?}}$$

$$B = \cancel{\sigma_1,}\ \sigma_2 \quad iu$$

$SO(2)$

<u>why ?</u>

$$\psi^T B \psi \to \psi^T U^T B U \psi =$$

$$= \psi^T B U^\dagger U \psi = \psi^T B \psi$$

$$\boxed{\begin{aligned} U^T B &= B U^\dagger \\ \Sigma^T B + B \Sigma &= 0 \end{aligned}}$$

$$\Downarrow$$

$$\sigma_2^T B + B \sigma_i = 0$$

$$\Downarrow$$

$$\boxed{B = \sigma_2}$$

$$\psi = \begin{pmatrix} u \\ d \end{pmatrix} \Rightarrow \psi^T i\sigma_2 \psi =$$

$$= ud - du$$

$$= \uparrow\downarrow - \downarrow\uparrow$$

$\downarrow$

Lorentz

$$\cdot\ \psi_L^T\, i\sigma_2\, C\, \psi_L \qquad \psi_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$$

$$= -\psi_L^T C^T (i\sigma_2)^T \psi_L$$

$$= -\psi_L^T(-C)(-i\sigma_2)\psi_L =$$

$$= -\psi_L^T C\, i\sigma_2 \psi_L = 0$$

$$\Downarrow$$

<u>no</u> mass term!?

$\boxed{\text{Is SU(2) chiral?}}$

$\boxed{\text{NO!}}$

$$\boxed{\psi_{1L}^T C\, i\sigma_2\, \psi_{2L} = -\psi_{2L}^T C\, i\sigma_2\, \psi_{1L}}$$

$$\boxed{\neq 0}$$

- with 1 gen. $\Rightarrow$ $SU(2)$ = chiral
- more gen. $\Rightarrow$ NOT chiral

  $\Rightarrow$ direct mass

different from $SO(2)$

$$\psi_1^T B \psi_2 = \psi_{1+} \psi_{2-} = u_1 v_2 \cdots$$

$$\Rightarrow \boxed{\psi_{1+}^T B \psi_{2+} = 0}$$

$\Rightarrow$ Truly chiral

$$\boxed{SO(4)}$$

$$\Gamma_i^{(4)} = \begin{pmatrix} 0 & \Gamma_i^{(3)} \\ \Gamma_i^{(3)} & 0 \end{pmatrix}, \quad \Gamma_4 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \qquad -\text{''}-$$

Lorentz $\updownarrow$

$$\gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \Gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Gamma_{FIVE} = (-i)^N \underbrace{\Gamma_1 \cdots \Gamma_{2N}}_{SO(2N)}$$

$$= (-1)\, \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4$$

$$= (-1)\begin{pmatrix} \sigma_1\sigma_2 & 0 \\ 0 & \sigma_1\sigma_2 \end{pmatrix}\begin{pmatrix} i\sigma_3 & 0 \\ 0 & -i\sigma_3 \end{pmatrix}$$

$$= \begin{pmatrix} +\mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}$$

$$\Downarrow$$

$$\boxed{\Gamma_\pm = \frac{1 \pm \Gamma_{FIVE}}{2}}$$

$$\Rightarrow \quad \psi_+ = \Gamma_+ \psi$$

$$\psi_- = \Gamma_- \psi$$

$$\Downarrow$$

$$\psi^T B \psi = \text{inv.}$$

$$SO(2) \Rightarrow \quad B = \sigma_1 \quad (\sigma_2)$$

$$\Downarrow$$

$$B = \Gamma_1 \Gamma_3 \Gamma_5 \cdots$$

or $$B = \Gamma_2 \Gamma_4 \Gamma_6 \cdots$$

$$B = -\sigma_1 \sigma_3 = \begin{pmatrix} +i\sigma_2 & 0 \\ 0 & +i\sigma_2 \end{pmatrix}$$

$$B = \Gamma_2 \Gamma_4 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}$$

$$B = \text{diagonal}$$

$$\Gamma_{FIVE} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Psi_+ = \begin{pmatrix} u \\ 0 \end{pmatrix} \therefore \Psi_+^T B \Psi_+ = \underline{\underline{u^T i\sigma_2 u}}$$

$$\Rightarrow \boxed{SO(4) = \text{vector-like}}$$

$$\boxed{SO(4) = SU(2) \times SU(2)}$$

$$\Gamma_{FIVE} = (-i)^N \Gamma_1 \cdots \Gamma_{2N}$$

$$\Sigma_{ij} = \frac{1}{4i} [\Gamma_i, \Gamma_j]$$

$$\Sigma_{12} = \frac{1}{2i} \Gamma_1 \Gamma_2, \quad \Sigma_{34} = \frac{1}{2i} \Gamma_3 \Gamma_4, \ldots$$

$$\Sigma_{2N-1,2N} = \frac{1}{2i} \Gamma_{2N-1,2N}$$

$$\Gamma_{FIVE} = (2\Sigma_{12})(2\Sigma_{34}) \cdots (2\Sigma_{2N-1,2N})$$

$$\boxed{\{\Sigma_{12}, \Sigma_{34}, \ldots \Sigma_{2N-1,2N}\} = \text{Cartan sub-algebra}}$$

$$\boxed{r(SO(2N)) = N}$$

$$2\Sigma_{12} = \frac{1}{i}\Gamma_1\Gamma_2 \Rightarrow (2\Sigma_{12})^2 = (-1)\Gamma_1\Gamma_2\Gamma_1\Gamma_2$$
$$= +1$$

$(2\Sigma_{2i-1,2i})$: ± 1 eigenvalues
$= \epsilon_i$ (± 1)

$$\psi_+ = \frac{1+\Gamma_{FIVE}}{2}\psi = \Gamma_+\psi$$

$$\boxed{\begin{array}{l}\Gamma_+\psi_+ = \psi_+ \Leftrightarrow \\ \Gamma_+ = 1 \text{ in } \psi_+ \text{ space}\end{array}}$$

$$\Rightarrow \Gamma_{FIVE}\psi_+ = \psi_+$$
$$\Gamma_{FIVE} = 1 \text{ in the sub-space}$$

$$\Gamma_{FIVE} = 2\Sigma_{12} \cdots 2\Sigma_{2N-1,2N}$$

$$\psi_+ = |\varepsilon_1 \cdots \varepsilon_N\rangle \;\therefore\; \left(\prod_{i=1}^{N} \varepsilon_i = 1\right)$$

<u>SO(2)</u> $\quad \psi_+ \text{ "=" } |+\rangle \text{ "=" } u \text{ "=" } \begin{pmatrix} u \\ 0 \end{pmatrix}$

<u>SO(4)</u> $\quad \psi_+ = |\varepsilon_1 \varepsilon_2\rangle \;\therefore\; \varepsilon_1 \varepsilon_2 = +1$

$$\Downarrow$$

$$\boxed{\begin{aligned} \psi_+ : \; & |++\rangle \\ & |--\rangle \end{aligned}} \qquad \{\Sigma_{12}, \Sigma_{34}\}$$

" Cartan

$$(T_3)_+ = \frac{1}{2}(\Sigma_{12} + \Sigma_{34}) \qquad (T_{3L})$$

$$(T_3)_- = \frac{1}{2}(\Sigma_{12} - \Sigma_{34}) \qquad (T_{3R})$$

$$\left.\begin{aligned}(T_3)_+ \,|++\rangle &= \tfrac{1}{2}\,|++\rangle \\ (T_3)_+ \,|--\rangle &= -\tfrac{1}{2}\,|++\rangle\end{aligned}\right\} SU(2)_+$$

$$\left.\begin{aligned}(T_3)_- \,|++\rangle &= 0 \\ (T_3)_- \,|--\rangle &= 0\end{aligned}\right\} \quad SU(2)_L$$

- $\psi_+ = |\varepsilon_1 \varepsilon_2\rangle \therefore \varepsilon_1 \varepsilon_2 = +1$

  = doublet of $SU(2)_+$
  singlet of $SU(2)_-$

- $\psi_- = |\varepsilon_1 \varepsilon_2\rangle \therefore \varepsilon_1 \varepsilon_2 = -1$

  $\hookrightarrow \begin{pmatrix} |+-\rangle \\ |-+\rangle \end{pmatrix}$ singlet of $SU(2)_+$
  doublet of $SU(2)_-$

$$\psi^T B \psi = \psi_+^T i\sigma_2 \psi_+ \; \cdots$$

$SO(4) \neq$ chiral (mass term)

(vector-like)

$$\boxed{SO(4) \longleftrightarrow \text{Lorentz}}$$

$$\psi_{1+,L}^T \, B \, C' \, \psi_{2+,L} =$$

$$= u_{1L}^T \, C \, i\sigma_2 \, u_{2L} =$$

$$= -u_{2L}^T \, C^T (i\sigma_2)^T u_{1L} = -u_{2L}^T \, C i\sigma_1 u_{1L}$$

$$\Rightarrow \boxed{u_{1L}^T C i\sigma_2 u_{1L} = 0}$$

(no mass for 1 gen.)

$\Downarrow$ more gen.

$SO(4)$ = bad group

= direct mass term

$SO(2)$: $\quad B = \sigma_1$ (off diagonal)

$SO(4)$ $\quad B = \begin{pmatrix} \pm i\sigma_2 & 0 \\ 0 & \pm i\sigma_2 \end{pmatrix}$ (diagonal)

$\Downarrow$

$SO(2)$ $\quad \psi_+ \psi_+ = 0$

$SO(4)$ $\quad \psi_+ \psi_+ \neq 0$

$\boxed{SO(4) = SU(2) \times SU(2)}$

$\Downarrow$

SO(5)

$$\left[\Gamma_i^{(4)}, \; \Gamma_{FIVE} = \Gamma_a^{(5)} \right] \quad i=1,\dots,4 \qquad a=1,\dots,5$$

$\boxed{SO(6)}$

$$(SO(6) = SU(4))$$

$$\boxed{G(PS) = SO(4)\times SO(6)}$$

$$\Gamma_i^{(6)} = \begin{pmatrix} 0 & \Gamma_i^{(5)} \\ \Gamma_i^{(5)} & 0 \end{pmatrix}, \qquad \Gamma_6^{(6)} = \begin{pmatrix} 0 & -i\mathbb{1}_4 \\ i\mathbb{1}_4 & 0 \end{pmatrix}$$

$i=1,\dots,5$

$$\Gamma_{FIVE} = (-1)^3 \Gamma_1 \cdots \Gamma_6 = \begin{pmatrix} +\mathbb{1}_4 & 0 \\ 0 & -\mathbb{1}_4 \end{pmatrix}$$

$$\Downarrow$$

$\psi^T B \psi = \text{inv.}$ $\quad B = \Gamma_1 \Gamma_3 \Gamma_5$

$(\Gamma_2 \Gamma_4 \Gamma_6)$

$$\left.\begin{array}{l} \Gamma_1 \, (\sigma_1) \\ \Gamma_3 \, (\sigma_3) \\ \Gamma_5 \, (1) \end{array}\right\} \Rightarrow \underbrace{\Gamma_1 \Gamma_3 \Gamma_5}_{\boxed{\text{off-diagonal}}} = \begin{pmatrix} 0 & f(\sigma_2) \\ f(\sigma_2) & \end{pmatrix}$$

$$\Rightarrow \boxed{\psi_+^T B \psi_+ = 0}$$

$$\Downarrow$$

$$\boxed{\begin{array}{c} SO(6) \sim SO(2) \\ \text{chiral} \end{array}}$$

$$\Downarrow$$

~~$SO(8) \Rightarrow B = $ diagonal~~ (4 off diag)

$$\boxed{SO(10) \Rightarrow B = \text{off-diagonal} \quad (5 \text{ -||- })}$$

good

$SO(2)$: $2^1 = 2^{2/2} = 2^{N/2}$

$$2 = 1_+ + 1_-$$

$SO(4)$: $4 = 2^2 = 2^{N/2}$

$$4 = 2_{(+)} + 2_{(-)}$$

$\boxed{SO(6)}$ $\boxed{15 = \frac{6 \cdot 5}{2}}$ $8 = 2^3 = 2^{N/2}$

$\boxed{r = 3}$

$\Vert$

$$\boxed{8 = 4_{(+)} + 4_{(-)}}$$

$\Uparrow$

$\boxed{SU(4)}$ $n=3$ Irreducible

$(15 \text{ gen} = 4^2 - 1)$

$$\boxed{d(4) = 2^N \text{ in } SO(2N)}$$

$$\boxed{SO(1, 2N-1)}$$