Lecture XII

11/12/2020

SU(5): keep building

$$5_w = \begin{pmatrix} 1 \\ 2 \\ 3 \\ \hdashline 4 \\ 5 \end{pmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix}1\\2\\3\end{matrix}}\right\} \text{color} \\ \left.\vphantom{\begin{matrix}4\\5\end{matrix}}\right\} \text{flavor} \end{array}$$

glues

$$U(1) \times \underbrace{SU(3)}_{r=2} \times \underbrace{SU(2)}_{+\;1} \leq \underbrace{SU(5)}_{r=4}$$

$$\Downarrow$$

$$\boxed{Y \propto \begin{array}{c} \text{color} \\ \left(\begin{array}{c:c} -2\,\mathbb{1}_3 & \\ \hdashline & 3\,\mathbb{1}_2 \end{array}\right) \\ \text{flavor} \end{array}}$$

$$\begin{cases} T_3 = T_3^{c} = \dfrac{1}{2}\begin{pmatrix} 1 & & & & \\ & -1 & & & 0 \\ & & 0 & & \\ 0 & & & 0 & \\ & & & & 0 \end{pmatrix} \\[2ex] T_8 = T_8^{c} = \dfrac{1}{2\sqrt{3}}\begin{pmatrix} 1 & & & & \\ & 1 & & & 0 \\ & & -2 & & \\ 0 & & & 0 & \\ & & & & 0 \end{pmatrix} \end{cases}$$

$$T_{23} = T_3^{w} = \frac{1}{2}\begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix}$$

no weak

$$(q^c)_L \equiv C\,\bar{q}_R^{\,T}$$

$$(\bar{5})_L = \begin{pmatrix} \bullet\ r \\ \bullet\ y \\ \bullet\ b \\ \hdashline \nu \\ e \end{pmatrix}_L \leftarrow \text{place ??}$$

$\}$ ℓ doublet

no color

$$\boxed{Q_{em} \in \{\text{Cartan}\}}$$

$$\mathrm{Tr}\, Q_{em} = 0$$

$$\underset{\substack{\shortparallel \\ 0}}{Q_\nu} + \underset{\substack{\shortparallel \\ -1}}{Q_e} + 3\, Q_{d^c} = 0$$

$$\Downarrow$$

$$Q_{d^c} = 1/3$$

2 miracles of nature

(i) $q = \pm n\, q_d$ $\longrightarrow$ positive integer

(ii) $q_\nu = 0$ $\quad (q_f = Q_f)$

$$\Downarrow$$

$$Q_\nu + Q_e + 3\,Q_{d^c} = 0$$

$$Q_\nu = Q_e + 1$$

$$Q = T_3 + (\alpha\, Y)$$

$$\Delta Q\ (\text{doublet}) = \Delta T_3\ (\text{doublet})$$
$$= +1$$

$$\boxed{\bar{5}_L = \left(\begin{array}{c} (d^c) \\ \hline \nu \\ e \end{array}\right)_L}$$

$$Q = T_3 + \cdots$$

$$\boxed{Q_u = Q_d + 1}$$

$$\Rightarrow \quad 5 = \left(\begin{array}{c} d^r \\ d^y \\ d^b \\ \hline e^c \\ -\nu^c \end{array}\right)_R \leftarrow \boxed{Y > 0}$$

weak

$$(e^c)_R = C\,\bar{e}_L^{\,T}$$

weak

$$D = \begin{pmatrix} \nu \\ e \end{pmatrix} \Rightarrow \boxed{i\sigma_2 D^* = \text{doublet}}$$

$$Y \propto \left(\begin{array}{c|c} -2\,\mathbb{1}_3 & \\ \hline & 3\,\mathbb{1}_2 \end{array}\right)$$

15 SM fermions

$$\left[\begin{array}{cc} \begin{pmatrix} u_L^\alpha \\ d_L^\alpha \end{pmatrix} & u_R^\alpha,\ d_R^\alpha \\ & 15 \\ \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} & e_R \end{array}\right]$$

in order to complete

$\Rightarrow$ find a 10 dim. repr.

$$(5\times 5) = \underbrace{\text{Sym}}_{(15)} + \underbrace{\text{Anti-sym}}_{(10)}$$

$\boxed{\text{but still, let's look at } 5}$

$$\begin{pmatrix} d^r \\ d^y \\ d^b \\ e^c \\ -\nu^c \end{pmatrix}_R$$

gluons

gluons

W

$X^{\alpha=r,y,b}$

$Y^{\alpha=r,y,b}$

$$u^\alpha = (u^r, u^y, u^b)$$

up quarks

X (3 colors), Y (3 colors)

(X^*), $\bar{X}$ (3 anti-colors), $\bar{Y}$ (3 anti-colors)

$$\# \text{ of states} = \underset{x}{3} + \underset{y}{3} + \underset{\bar{x}}{3} + \underset{\bar{y}}{3} = 12$$

$$\boxed{2 \text{ particles} = \underbrace{(x, y)}_{\text{SU(2) doublet, SU(3) triplet (flavor + color)}}}$$

$$SU(5): \# \text{ of gen} = 5^2 - 1 = 24$$

$\Rightarrow$ 24 gauge bosons

$$D_\mu = \partial_\mu - ig A_\mu^a T^a \qquad a = 1, \ldots, 24$$

SM: $SU(3) \times SU(2) \times U(1)$

$$\underbrace{8\,\text{gen.} + 3\,\text{gen.} + 1\,\text{gen}}_{12\,\text{gen.} \Rightarrow 12\,\text{gauge bosons}}$$

$$\boxed{8\ \text{gluons},\ W^+,\ W^-,\ Z,\ \gamma}$$

$$\boxed{24 = \underbrace{12}_{SM} + \underbrace{12}_{(X,Y)}}$$

$\Downarrow$ $\boxed{\text{interactions}}$

$$\mathcal{L} = i\,\bar{\Psi}\gamma^\mu D_\mu \Psi$$

$$\Psi = 5_R\,(F)$$

$$5_R = \begin{pmatrix} d^\alpha \\ \hdashline e^c \\ -\nu^c \end{pmatrix}_R$$

color singlet

$$X_\alpha\, \overline{e^c_R}\, \gamma^\mu d^\alpha_R \leftarrow \text{color triplet}$$

color triplet

$$Y_\alpha\, \overline{\nu^c_R}\, \gamma^\mu d^\alpha_R$$

$$Q_X = 4/3$$

$$Q_Y = 1/3$$

$$\begin{pmatrix} X \\ Y \end{pmatrix}$$

$$\boxed{\Delta Q\ (\text{doublet}) = 1}$$

[X, Y = color triplets since they transform a quark (d) into leptons]

$$\underbrace{\bar{q}_\alpha (\text{glue})^\alpha_{\ \beta}\, q^\beta}$$

$$3 \times 3^* = 8 + 1$$

8 gluons $\Leftrightarrow$

$\Leftrightarrow$ taking q into ℓ

$$X\,[\ \overline{e^c}\quad d\ ]$$

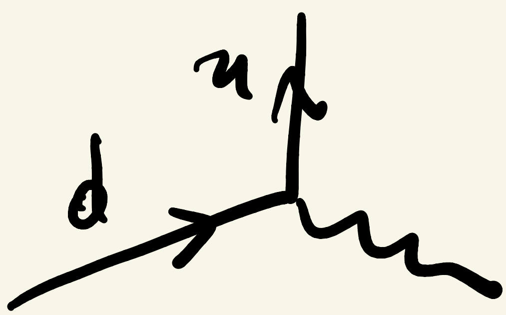
X = the same quantum # as q

= the same -11- # of l

$$\Downarrow$$

$$\boxed{\text{lepto-quark}}$$

$$W\{\underbrace{[\bar{q}\ q]}_{\text{carries no quantum \# of } q} + [\bar{l}\ e]\}$$



$$\begin{pmatrix} x \\ y \end{pmatrix}^{\alpha} \leftarrow W \qquad \begin{pmatrix} \nu \\ e \end{pmatrix} \leftarrow W$$

$\begin{pmatrix} x \\ y \end{pmatrix}^{\alpha} \Longleftrightarrow$ leptons

$\Updownarrow$ quarks $\Longrightarrow$ $\boxed{\text{lepto-quarks}}$

$$\boxed{5_R = \begin{pmatrix} d \\ e^c \\ -\nu^c \end{pmatrix}_R}$$

$\sim$

$$\bar{5}_L = \begin{pmatrix} (d^c) \\ \text{- - - -} \\ \nu \\ e \end{pmatrix}_L$$

$\bar{X} \; \bar{e} \; d^c$

$\boxed{X \longleftrightarrow e}$

$L \quad y \leftrightarrow \nu$

$$\boxed{\text{we are } \underline{\text{not}} \text{ done}}$$

$$15 = \bar{5} + \cancel{\bar{5} + \bar{5}}$$

$\downarrow$

(ν, e, d^c) **not enough**

$\Downarrow$

$$\boxed{15 = \boxed{\bar{5}_L + 10_L} \quad (5_R + 10_L)}$$

$\cancel{E = ma^2}$

$\cancel{E = mb^2}$

$$E = mc^2$$

$$\boxed{10 = \text{anti-symmetric}}$$

SU(2) flavor

color

$$\left(\begin{array}{ccc|cc} 0 & u^c & u^c & u^r & d^r \\ & 0 & u^c & u^y & d^y \\ & & 0 & u^b & d^b \\ \hline -u^r & -u^y & -u^b & 0 & e^c \\ -d^r & -d^y & -d^b & -e^c & 0 \end{array}\right)_L$$

color

anti-quark

$$\boxed{(q^c)_L = C\,\bar{q}_R^{\,T}}$$

$$\boxed{15 = \bar{5} + 10}$$

$\parallel$

SM

old fermions

$$(\underbrace{5 \times 5}_{})_{AS} = A_{ij}$$

$$A_{ij} \sim 5_i \, 5_j$$

$$A_{ij} \to U_{ik} \, 5_k \, U_{jl} \, 5_l =$$

$$= U_{ik} \, A_{kl} \, U_{jl}$$

$$= U_{ik} \, A_{kl} \, U^T_{lj}$$

$$= (U A U^T)_{ij}$$

$$= (1 + i\theta_a T_a) \, A \, (1 + i\theta_a T_a)^T$$

$$= A + i\theta_a (T_a A + A T_a^T)$$

$$\hat{T}_a A = T_a A + A T_a^T$$

$$\frac{\lambda_a}{2} \quad a = 1, \ldots, 24$$

$\sigma_{1,2}$ in all off-diagonal (20)

+ 4 Cartan

$$\boxed{T_{3c},\ T_{8c},\ T_{3w},\ Y}$$

$Q \in SU(5)$

$$\Rightarrow \boxed{Q = c_a T_a}$$

$$\boxed{A = \mathbb{1}0}$$

$$\boxed{\hat{Q}\,\mathbb{1}0 = Q\,\mathbb{1}0 + \mathbb{1}0\,Q^T}$$

$$\boxed{Q = \text{diagonal}}$$

$$\boxed{(\hat{Q}\,\mathbb{1}0)_{ij} = (q_i\,\mathbb{1}0_{ij} + \mathbb{1}0_{ij}\,q_j)}$$

$$(\hat{Q}\,5)_i = (Q\,5)_i = q_i\,5_i$$

$$Q(5) = \begin{pmatrix} -1/3 & & & & \\ & -1/3 & & & \\ & & -1/3 & & \\ & & & 1 & \\ & & & & 0 \end{pmatrix}$$

$$q_i = (-1/3, -1/3, -1/3, 1, 0)$$

$$\boxed{(Q\,10)_{ij} = (Q_i + Q_j)\,10_{ij}}$$

$$5 = \begin{pmatrix} d \\ \hline e^c \\ \nu^c \end{pmatrix} \qquad d \leftarrow Q_i = -1/3, \quad e^c \rightarrow 1, \quad \nu^c \rightarrow$$

$Q(10)_{14} =$

$= Q_1 + Q_4$

$= -\frac{1}{3} + 1$

$= 2/3$

$= Q_d + Q_{e^c}$

14

$$10 = \begin{pmatrix} 0 & x & x & u & d \\ & 0 & x & & \bullet \\ & & 0 & & \bullet \\ & & & & \end{pmatrix}$$

$Q = -2/3$

e^c

$\cancel{Q_{e^c}} = Q_4 + Q_5 =$

$= \cancel{Q_{e^c}} + Q_{\nu^c}$

$d = 15$

$\Downarrow$

$Q_d = Q_1 + Q_5$

$Q_{u^c} = 2\,Q_d$

$Q_u = Q_d + Q_{u^c}$

$Q_d = Q_d + Q_{\nu^c}$

$$\boxed{Q_{e^c} = 1} \Longleftrightarrow \boxed{Q_{\nu^c} = 0} \Rightarrow \boxed{Q_\nu = 0}$$

$$\sum_Q (\text{in } 5) = 0$$

$$\text{Tr}\, Q = 0$$

$$\Downarrow$$

$$3Q_d + Q_{e^c} + \cancel{Q_{\nu^c}} = 0$$

$$\Downarrow$$

$$\boxed{Q_{e^c} = -3Q_d} \Rightarrow \text{[illegible]}$$

$$\Downarrow$$

$$Q_d = -1/3$$

$\boxed{\text{SM}}$

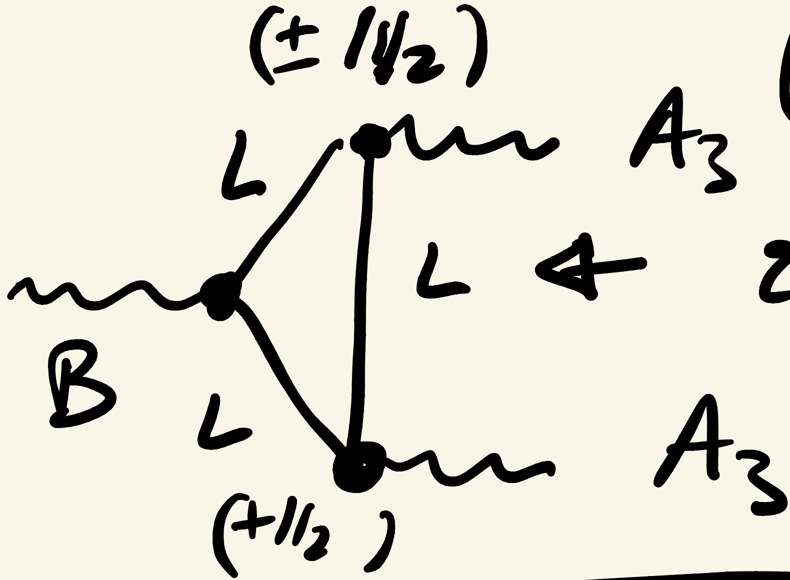
$$l = \begin{pmatrix} \nu \\ e \end{pmatrix}_L \qquad e_R \, , \left(e^c = C \bar{e}_R^T \right)$$

the same electron

$Q = T_3 + \frac{Y}{2}$ ← arbitrary

$$\Rightarrow \boxed{\begin{aligned} Q_{e_L} &= -\tfrac{1}{2} + Y_\ell/2 \\ &\ \| \\ Q_{e_R} &= Y_{e_R}/2 \end{aligned}}$$

$$\boxed{Y_{e_R} = Y_e - 1}$$



$(\pm \frac{1}{2})^2 = \frac{1}{4}$

$q = \begin{pmatrix} u \\ d \end{pmatrix}$

$\ell = \begin{pmatrix} \nu \\ e \end{pmatrix}_L$

$$\Rightarrow \boxed{\mathrm{Tr}\, Y_L = 0}$$

$\Downarrow$

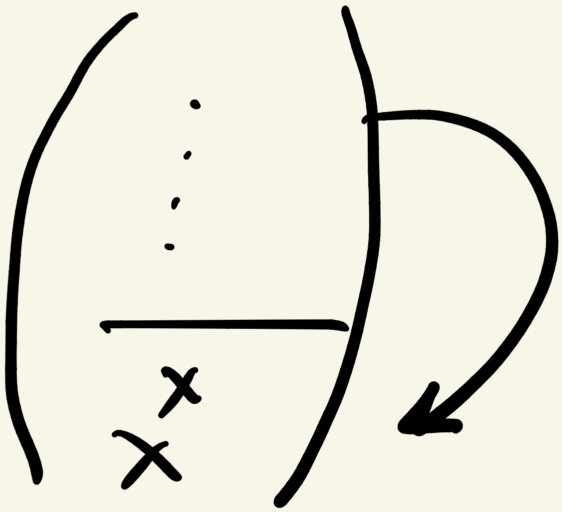
$$\boxed{3Y_q + Y_e = 0}$$

<u>SM</u> cannot <u>per se</u> fix Y
(or charge)
but with anomaly $= 0$
$\Downarrow$
charge is quantised

add new fermion F

$\boxed{\begin{array}{c} F_L \Longleftrightarrow F_R \\ \text{same all} \end{array}} \not\Rightarrow$ anomaly $= 0$

$$\boxed{Q_F = \text{arbitrary}}$$



SU(5)

connects

$\mathrm{Tr}\, Y = 0$

$\Downarrow$

$$\boxed{\text{all charges are quantized}}$$

if I find new F

$Q_F = -1/3$

$\Longrightarrow$ put it in 5 $(\bar{5})$

$\Downarrow$

$\exists\, F'$: $\boxed{Q_{F'} = (1,0)}$

- you find: $\underbrace{F''_L,\ F''_R}_{\text{same repr.}}$

$Q_{F''} = 3.25641$

$\Rightarrow$ SU(5) ruled out

Minimal SU(5) $\Leftrightarrow$

SM

for neutrino

both are incomplete

$$10 = \begin{array}{c} \xrightarrow{\quad} \quad \underrightarrow{\text{weak}} \\ \left(\begin{array}{c|cc} u^c & u & d \\ \hline & & e^c \end{array} \right) \end{array} \downarrow \text{color}$$

$u^c \to u$

$u^c \to d$

(*)

$X\,[\bar{e}^c d]$

lepto-quarks

$[\bar{u}^{\alpha} u^{c\beta}]\, X^{\gamma}$ (**) $\epsilon_{\alpha\beta\gamma}$

$[\bar{d}\, u^c]\, Y$

di-quarks

lepton-quark: you always

take $l \to q$

l, q are conserved

~~lepton-quark~~ $\Rightarrow$

fixed L, B #

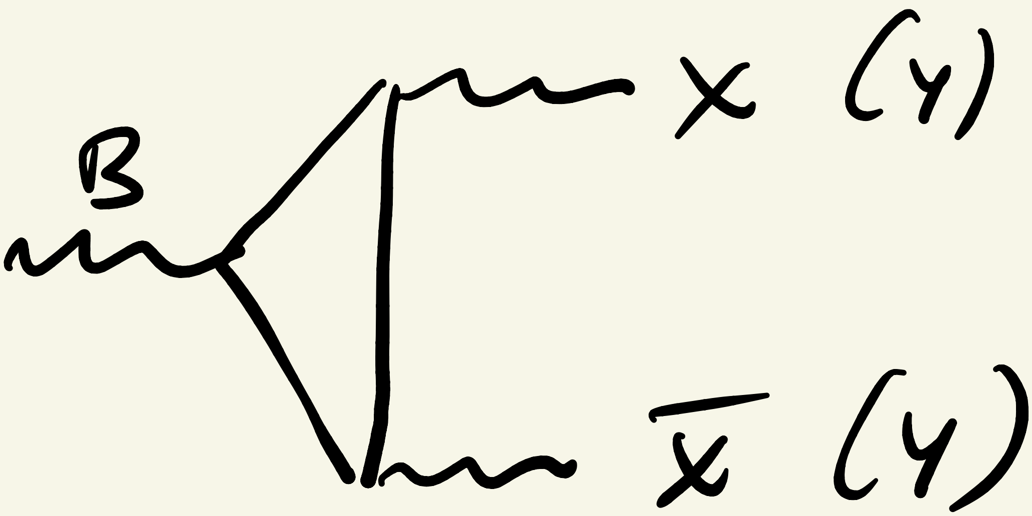
SU(5): $\boxed{\text{Charge is quantised}}$

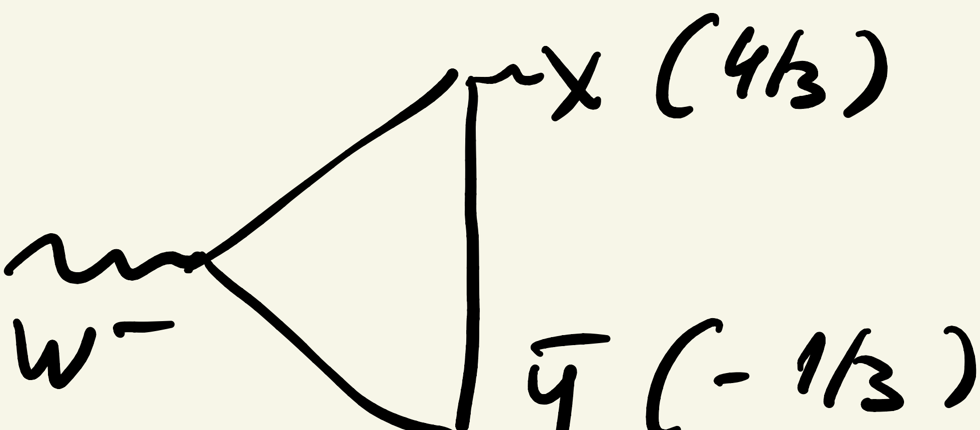
$\Downarrow$

$\boxed{\text{SM}}$ anomaly $= 0$

$\Rightarrow$ Q is quantised

$$\boxed{\text{Anomaly} = 0\ ?}$$





$$= 0$$

$$\boxed{\underline{\underline{\mathcal{A}(R)}}\, d_{abc} = \mathrm{Tr}_{(R)} \{T_a, T_b\} T_c}$$

$$\boxed{\mathcal{A}(F) = \mathcal{A}(5) = 1}$$

$$\mathcal{A}(10) = \mathcal{A}(\text{anti-sym})$$

$$= \{ N-4 \text{ for } SU(N) \}$$

$$= \underline{\underline{1 \text{ for } SU(5)}}$$

$$\boxed{\bar{5}_L, \quad 10_L}$$

$\mathcal{A}(\bar{5}) = -1 \quad \| \quad \mathcal{A}(10) = +1$

$$5 \to e^{i\theta T}\, 5$$

$$5^* \to e^{-i\theta T^*}\, 5^*$$

$$\hat{T}(5^*) = -T^* = -T^T$$

$$T^\dagger = T$$

$$\mathrm{Tr}\{T,T\}T = \mathrm{Tr}\{T^T,T^T\}T^T$$

$$\Rightarrow \quad A(5^*) = -1$$

$$\Downarrow$$

$$\boxed{A(\bar{5}_L + 10_L) = 0}$$

Q.E.D.

$\boxed{\text{check exactly to one generator}}$

$$Y \propto \begin{pmatrix} -2\,\mathbb{1}_3 & \\ & 3\,\mathbb{1}_2 \end{pmatrix}_{(5)}$$

$$\Downarrow$$

$$\mathrm{Tr}\, Y^3_{(5)} = (-2)^3 \cdot 3 + 3^3 \cdot 2$$

$$= [-8 \cdot 3 + 9 \cdot 3 \cdot 2]$$

$$= 3\,[-8 + 18] = 30$$

$$\boxed{\mathrm{Tr}\, Y^3_{(\bar{5})} = -30}$$

$\boxed{10}$

$$\left(\begin{array}{c|c} y= & \\ -2-2= & y=-2+3 \\ =-4 & =1 \\ \hline & \\ & ec\ (45) \\ & y=3+3=6 \end{array}\right)$$

$$Tr\ Y_{10}^{3} = (-4)^3 \cdot 3 + 1^3 \cdot 3 \cdot 2 + 6^3$$

$$= [-64 \cdot 3 + 72 \cdot 3 + 6]$$

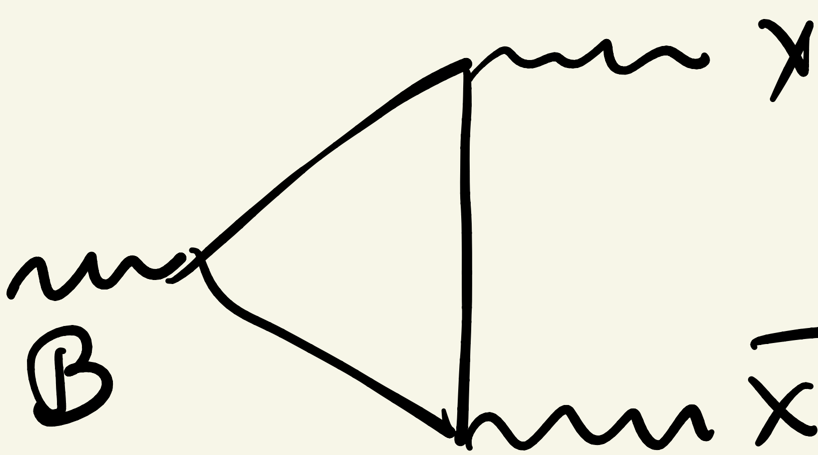
$$= [8 \cdot 3 + 6] = 30_w$$

$$\Rightarrow \boxed{VA(10_2) = 30}$$

$$\boxed{\begin{aligned} A(SU(5)) &= A(\bar{5}_L) + A(10) \\ &= -30 + 30 = 0 \end{aligned}}$$

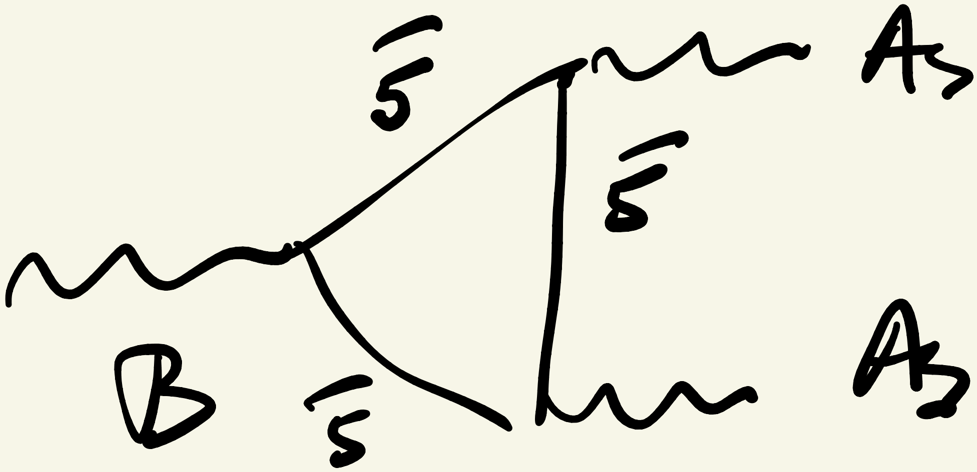
$$\cancel{SU(2)_L \times SU(2)_R \times U(1)}$$

$$\cancel{\begin{pmatrix} \nu \\ e \end{pmatrix}_L \quad \begin{pmatrix} \nu \\ e \end{pmatrix}_R \quad \bar{g}}$$



$= 0$

Anomaly = overall



$$= 0$$

(SM)

$$\boxed{\bar{\Psi}_{(R)} \gamma^\mu D_\mu \Psi_{(R)}}$$

$$\boxed{\mathcal{A}(R_1 \oplus R_2) = \mathcal{A}(R_1) + \mathcal{A}(R_2)}$$