Problem Set 5:

Handout: Thu, Jan. 14, 2021; Solutions: Fri, Jan. 22, 2021

Problem 1 Flux insertion (for the proof of Luttinger's theorem) – counts as two [a-d / e-g]

In this problem we consider non-interacting spin-less particles on a ring (or a torus) and couple them to a $U(1)$ gauge flux through the ring (or through one cycle of the torus).

(1.a) We start by considering a one-dimensional lattice on a ring of length L. The simplest way to introduce a total $U(1)$ gauge flux Φ is by the Hamiltonian:

$$\hat{\mathcal{H}}(\Phi) = -t \sum_{j=1}^{L-1} \left(\hat{c}^\dagger_{j+1} \hat{c}_j + \text{h.c.} \right) - t \left(e^{i\Phi} \hat{c}^\dagger_1 \hat{c}_L + e^{-i\Phi} \hat{c}^\dagger_L \hat{c}_1 \right). \tag{1}$$

For which values of Φ is this Hamiltonian translationally invariant? How are eigenstates at $\Phi =$ and $\Phi = 2\pi$ related to one another?

(1.b) Find a unitary gauge transformation

$$\hat{U} = \exp\left[-i \sum_{j=1}^{L} \varphi_j \hat{n}_j \right] \tag{2}$$

such that

$$\tilde{\mathcal{H}}(\Phi) = \hat{U}^\dagger \hat{\mathcal{H}}(\Phi) \hat{U} \tag{3}$$

is translationally invariant (make an appropriate choice of φ_j and calculate $\tilde{\mathcal{H}}(\Phi)$ explicitly!). How are eigenstates at $\Phi = 0$ and $\Phi = 2\pi$ related?

(1.c) Using Fourier transformations, derive all eigenenergies $E_n(\Phi)$ of $\tilde{\mathcal{H}}(\Phi)$ for general values of Φ. Show that the corresponding eigenstates are plane waves with momentum

$$k_n = \frac{2\pi}{L} n, \qquad n = 1 ... L, \tag{4}$$

and show that eigenenergies are related as:

$$E_n(\Phi + 2\pi) = E_{n+1}(\Phi). \tag{5}$$

(1.d) Now consider an initial eigenstate $|\Psi_0(\Phi)\rangle$ of $\tilde{\mathcal{H}}(\Phi)$ for $\Phi = 0$ with N particles with momenta k_{n_m}, where $m = 1 ... N$ labels the particles and $n_m \in \{1, 2, ..., L\}$. Express the total momentum P_x of this state in terms of the momenta k_{n_m}.

(1.e) Next, assume that Φ is adiabatically increased from $\Phi = 0$ to $\Phi = 2\pi$, such that the quantum numbers k_{n_m} cannot change. Accordingly, P_x cannot change. Show that the new eigenstate $|\Psi_1\rangle = |\Psi_0(\Phi = 2\pi)\rangle$ of $\tilde{\mathcal{H}}(\Phi = 2\pi)$ is related to $|\Psi_0(\Phi = 0)\rangle$ by a gauge transformation $\hat{V}$:

$$|\Psi_1\rangle = \hat{V}^\dagger |\Psi_0(\Phi = 0)\rangle, \qquad \hat{V} = \exp\left[-i \sum_{j=1}^{L} \vartheta_j \hat{n}_j\right] \tag{6}$$

for appropriately chosen values of ϑ_j. *Hint:* Show that $\tilde{\mathcal{H}}(\Phi = 2\pi)$ and $\tilde{\mathcal{H}}(\Phi = 0)$ are related by the gauge transformation $\hat{V}$.

(1.f) Show that $|\Psi_1\rangle$ is also an eigenstate of $\tilde{\mathcal{H}}(\Phi = 0)$ but with momentum:

$$P'_x = P_x + \frac{2\pi}{L} N \mod 2\pi. \tag{7}$$

Hint: Use the relation from (1.d).

(1.g) Generalize your results from above for a higher-dimensional system on a $L_x \times L_y$ torus and show that

$$P'_x = P_x + \frac{2\pi}{L_x} N \mod 2\pi \tag{8}$$

when flux Φ_x is adiabatically introduced through the x-cycle of the torus. Here N still denotes the total particle number in the higher-dimensional system.

Problem 2 Small and large Fermi surfaces in the Hubbard model

(2.a) When a translational symmetry is spontaneously broken, Luttinger's theorem can be applied for the resulting reduced Brillouin zone. Show for the case of a Néel state, i.e. for a square lattice with a broken sub-lattice symmetry, that Luttinger's theorem in the reduced magnetic Brillouin zone (MBZ) becomes:

$$\frac{V_{\rm FS}^{\rm MBZ}}{2\pi^2} = \mathbb{Z} - p \tag{9}$$

where p denotes the hole doping, i.e.

$$N = L_x L_y (1 - p). \tag{10}$$

Consider a spin-balanced system where $N_\uparrow = N_\downarrow = N/2$.

(2.b) Perform a particle-hole mapping, $\hat{c}_{\boldsymbol{j},\sigma} \to \hat{h}^\dagger_{\boldsymbol{j},\sigma}$, and show Luttinger's theorem formulated for the hole-fermi surface becomes:

$$\frac{V^h_{\rm FS}}{2\pi^2} \equiv \frac{N_h}{L_x L_y} \mod 2. \tag{11}$$

Here N_h denotes the number of (spin-full) holes and $V^h_{\rm FS} = (2\pi)^2 - V_{\rm FS}$.

(2.c) Combine your results from (a) and (b) to show that:

$$\frac{V^h_{\rm FS}}{2\pi^2} \equiv 1 + p \mod 2 \qquad \text{large FS} \tag{12}$$

for translationally invariant systems, and

$$\frac{V^h_{\rm FS}}{2\pi^2} \equiv p \mod 1 \qquad \text{small FS} \tag{13}$$

for a broken translational symmetry in the case of a Néel state.