Problem Set 4:

Handout: Thu, Dec. 17, 2020; Solutions: Fri, Jan. 8, 2020

Problem 1 Collapse and revival of a BEC

We consider the collapse and revival experiments in an optical lattice [Greiner et al., Nature 419 (2002)]. They start from an initial weakly interacting superfluid state, which can be modeled by a product of coherent states:

$$|\Psi(t=0)\rangle = \prod_{\boldsymbol{j}} |\alpha\rangle_{\boldsymbol{j}}, \qquad |\alpha\rangle_{\boldsymbol{i}} = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \tag{1}$$

At time $t=0$, the depth of the optical lattice is suddenly increased, leading to very strong on-site Hubbard interactions. At times $t>0$ we can model the Hamiltonian by just the interacting part

$$\hat{\mathcal{H}}_U = \frac{U}{2} \sum_{\boldsymbol{j}} \hat{n}_{\boldsymbol{j}} \left(\hat{n}_{\boldsymbol{j}} - 1\right), \qquad \text{with :} \quad \hat{n}_{\boldsymbol{j}} = \hat{\Psi}^\dagger_{\boldsymbol{j}} \hat{\Psi}_{\boldsymbol{j}}. \tag{2}$$

In the following you may assume that the density is small:

$$N/L^d \gg 1. \tag{3}$$

(1.a) How is $\alpha \in \mathbb{C}$ related to the average particle number N/L^d in the lattice? What is the physical meaning of the complex phase of α?

(1.b) After a time-evolution for a period T with $\hat{\mathcal{H}}_U$, the optical lattice is released and the system undergoes a time-of-flight (TOF) expansion. Show that the amplitude of interference peaks in the TOF images first *collapses* and later *revives* for special values $T_n = nT_0$ of T. Determine the revival time T_0! *Hint:* You may already use the result from (1.c) that $\langle \hat{\Psi}_{\boldsymbol{j}} \rangle \approx 0$.

(1.c) Show that half-way between the revivals the system realizes a so-called *cat state*, in which $\langle \hat{\Psi}_{\boldsymbol{j}} \rangle \approx 0$ but $\langle \hat{\Psi}^2_{\boldsymbol{j}} \rangle \neq 0$.

Problem 2 Generalized RPA calculation

Consider the Fermi-Hubbard model with nearest-neighbor tunneling t and on-site interactions U between the two spin states $\sigma = \uparrow, \downarrow$.

(2.a) Using the Hartree-Fock decoupling, derive the equations of motion for

$$\hat{\rho}_{p,q,\sigma} = \hat{c}^\dagger_{p+q,\sigma} \hat{c}_{p,\sigma} \tag{4}$$

in the generalized RPA approximation (as defined in the lecture notes) – i.e. include direct and exchange terms and approximate

$$\langle \hat{c}^\dagger_{p+q,\sigma} \hat{c}_{p,\sigma'} \rangle \approx \delta_{q,0} \delta_{\sigma,\sigma'} n^{\mathrm{F}}_p(T). \tag{5}$$

(2.b) From the equations of motion in (2.a) [the result can also be found in the lecture notes] derive the generalized RPA (gRPA) response functions for charge and spin:

$$\chi^{\mathrm{c}}_{\mathrm{gRPA}}(q,\omega) = \frac{\chi_0(q,\omega)}{1-\chi_0(q,\omega)U/L^d}, \qquad \chi^{\mathrm{s}}_{\mathrm{gRPA}}(q,\omega) = \frac{\chi_0(q,\omega)}{1+\chi_0(q,\omega)U/L^d}. \tag{6}$$

Problem 3 Super-exchange in a tilted potential

Consider a two-site (pseudo) spin-$1/2$ Hubbard model in the presence of a strong tilt Δ:

$$\hat{\mathcal{H}} = -t \sum_{\sigma=\uparrow,\downarrow} \left(\hat{c}^\dagger_{2,\sigma}\hat{c}_{1,\sigma} + \text{h.c.} \right) + \frac{\Delta}{2} \sum_{\sigma=\uparrow,\downarrow} \left(\hat{c}^\dagger_{1,\sigma}\hat{c}_{1,\sigma} - \hat{c}^\dagger_{2,\sigma}\hat{c}_{2,\sigma} \right) + \\ + U \sum_{n=1}^{2} \hat{n}_{n,\uparrow}\hat{n}_{n,\downarrow} + \frac{U}{2} \sum_{n=1}^{2} \sum_{\sigma=\uparrow\downarrow} \hat{n}_{n,\sigma}\left(\hat{n}_{n,\sigma}-1\right) \tag{7}$$

Note that the last sum is identical zero for fermionic particles (Pauli principle). In this problem we will consider the regime:

$$U,\Delta,t > 0, \qquad U,\Delta \gg t, \qquad |U \pm \Delta| \gg t, \tag{8}$$

and we work with *exactly two particles with arbitrary spin*.

(3.a) In the following we will treat t as a perturbation and work with unperturbed states with *exactly one particle per site*. Discuss for which values of U,Δ such states are ground states when $t=0$, or metastable excited states respectively.

(3.b) Now consider the particles $\hat{c}_{j,\sigma}$ are *fermions*. Show by an explicit perturbative calculation (degenerate perturbation theory) that the perturbed eigenstates for $t>0$ are described by the effective Hamiltonian

$$\hat{\mathcal{H}}_{\text{eff}} = J\hat{\boldsymbol{S}}_1\cdot\hat{\boldsymbol{S}}_2 - \frac{J}{4}\hat{n}_1\hat{n}_2, \tag{9}$$

where $\hat{\boldsymbol{S}}_n$ and $\hat{n}_n$ are the spin and particle number on site $n=1,2$. Show that the super-exchange coupling is given by

$$J = \frac{2t^2}{U+\Delta} + \frac{2t^2}{U-\Delta} \tag{10}$$

and discuss under which conditions it is (anti-) ferromagnetic.

(3.c) Next, assume that the particles $\hat{c}_{j,\sigma}$ are *bosons*. How do the results from (3.b) change?

(3.d) Finally, discuss how the situation changes if $\hat{c}_{j,\sigma}$ are *bosons* and the Hubbard interaction becomes spin-dependent, i.e.

$$\hat{\mathcal{H}}_U = U_{\uparrow\downarrow} \sum_{n=1}^{2} \hat{n}_{n,\uparrow}\hat{n}_{n,\downarrow} + \frac{1}{2} \sum_{n=1}^{2} \sum_{\sigma=\uparrow\downarrow} U_\sigma \hat{n}_{n,\sigma}\left(\hat{n}_{n,\sigma}-1\right) \tag{11}$$

instead of the SU(2)-invariant Hubbard interactions in Eq. (7). Hint: See e.g. Duan et al., Phys. Rev. Lett. 91, 090402 (2003).