Fakultät für Physik im WiSe 2020/21
Strongly correlated quantum systems
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https://www.physik.uni-muenchen.de/lehre/vorlesungen/wise_20_21/CorrQuantumSys/

Problem Set 1:

Handout: Thu, Nov. 5, 2020; Solutions: Fri, Nov. 13, 2020

Problem 1 Correlations in a classical ferromagnet

(1.a) Consider a classical ferromagnetic chain in 1D, with all spins pointing along z:

$$|\mathrm{FM}_z\rangle = |... \uparrow_{-1}\uparrow_0\uparrow_1 ...\rangle = \prod_j |\uparrow\rangle_j. \tag{1}$$

Calculate the following spin-spin two point correlation functions for distances d,

$$C^z(d) = \langle \hat{S}^z_0 \hat{S}^z_d\rangle, \qquad C^z_c(d) = \langle \hat{S}^z_0 \hat{S}^z_d\rangle_c, \tag{2}$$

and the reduced density matrix of the central spin $\hat{\rho}_0 = \mathrm{tr}_{j\neq 0}\left(|\mathrm{FM}_z\rangle\langle \mathrm{FM}_z|\right)$.

Is the pure state $|\mathrm{FM}_z\rangle$ entangled? Is the pure state $|\mathrm{FM}_z\rangle$ correlated?

(1.b) Consider a classical ensemble of ferromagnetic states $|\mathrm{FM}_{\boldsymbol{\Omega}}\rangle$ whose spins point in a random direction $\boldsymbol{\Omega}$ on the unit sphere. These states can be obtained from $|\mathrm{FM}_z\rangle$ by applying a global rotation $\hat{R}_{\boldsymbol{\Omega}}$ of all spins, which maps spins pointing along z to spins pointing along $\boldsymbol{\Omega}$, i.e. $|\uparrow\rangle \to \hat{R}_{\boldsymbol{\Omega}}|\uparrow\rangle$ with $\boldsymbol{\Omega}\cdot\hat{\boldsymbol{S}}\ \hat{R}_{\boldsymbol{\Omega}}|\uparrow\rangle = +1/2$. The classical ensemble can be described by a density matrix

$$\hat{\rho} = Z^{-1}\int d^2\boldsymbol{\Omega}\ \hat{R}_{\boldsymbol{\Omega}}|\mathrm{FM}_z\rangle\langle \mathrm{FM}_z|\hat{R}^\dagger_{\boldsymbol{\Omega}} \tag{3}$$

where Z serves as a normalization constant.

Calculate the following spin-spin two point correlation functions for distances d,

$$C^z(d) = \langle \hat{S}^z_0 \hat{S}^z_d\rangle, \qquad C^z_c(d) = \langle \hat{S}^z_0 \hat{S}^z_d\rangle_c, \tag{4}$$

and the reduced density matrix of the central spin $\hat{\rho}_0 = \mathrm{tr}_{j\neq 0}\left(\hat{\rho}\right)$.

Is the mixed state $\hat{\rho}$ correlated?

(1.c) Consider a classical ensemble of spins at infinite temperature, described by the density matrix

$$\hat{\rho}_\infty = \prod_j \frac{1}{2}\Big(|\uparrow\rangle_j\langle\uparrow| + |\downarrow\rangle_j\langle\downarrow|\Big). \tag{5}$$

Calculate the following spin-spin two point correlation functions for distances d,

$$C^z(d) = \langle \hat{S}^z_0 \hat{S}^z_d\rangle, \qquad C^z_c(d) = \langle \hat{S}^z_0 \hat{S}^z_d\rangle_c, \tag{6}$$

and the reduced density matrix of the central spin $\hat{\rho}_0 = \mathrm{tr}_{j\neq 0}\left(\hat{\rho}_\infty\right)$.

Is the mixed state $\hat{\rho}_\infty$ correlated?

Problem 2 Valence bond solid (VBS)

The ground state of the two-site antiferromagnetic ($J > 0$) Heisenberg spin Hamiltonian,

$$\hat{\mathcal{H}}_2 = J\hat{\boldsymbol{S}}_1 \cdot \hat{\boldsymbol{S}}_2, \tag{7}$$

is given by the spin-singlet, or Bell state

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right). \tag{8}$$

(2.a) Consider a one-dimensional spin chain with an antiferromagnetic Heisenberg term on every second bond:

$$\hat{\mathcal{H}}_{\mathrm{VBS}} = J\sum_n \hat{\boldsymbol{S}}_{2n-1} \cdot \hat{\boldsymbol{S}}_{2n}. \tag{9}$$

Find and describe the unique ground state $|\Psi_{\mathrm{VBS}}\rangle$ of $\hat{\mathcal{H}}_{\mathrm{VBS}}$. ($\hat{\mathcal{H}}_{\mathrm{VBS}}$ is called the parent Hamiltonian of this VBS state.)

(2.b) Assume that the system is realized in the ground state $|\Psi_{\mathrm{VBS}}\rangle$. Calculate the following spin-spin two point correlation functions for distances d,

$$C^z(d) = \langle \hat{S}^z_0 \hat{S}^z_d \rangle, \qquad C^z_c(d) = \langle \hat{S}^z_0 \hat{S}^z_d \rangle_c, \tag{10}$$

and the reduced density matrix of the central spin $\hat{\rho}_0 = \mathrm{tr}_{j\neq 0}\left(|\Psi_{\mathrm{VBS}}\rangle\langle\Psi_{\mathrm{VBS}}|\right)$.

Is the pure state $|\Psi_{\mathrm{VBS}}\rangle$ entangled? Is the pure state $|\Psi_{\mathrm{VBS}}\rangle$ correlated?

Problem 3 Entanglement area & volume laws

(3.a) Consider a two-dimensional valence bond solid $|\mathrm{VBS}\rangle$, determined as the ground state of the parent Hamiltonian

$$\hat{\mathcal{H}}^{\mathrm{2d}}_{\mathrm{VBS}} = J\sum_n \sum_{j_y} \hat{\boldsymbol{S}}_{2n-1,j_y} \cdot \hat{\boldsymbol{S}}_{2n,j_y}. \tag{11}$$

Here $\hat{\boldsymbol{S}}_{j_x,j_y}$ denotes the spin operator on site (j_x, j_y) in a 2D square lattice.

Let A be a rectangular region of size $\ell \times \ell$ in the center of the system. Calculate the entanglement entropy S_{A} of the subsystem. Distinguish between different possible sizes and locations of the rectangular region A relative to the 2D lattice! How does S_{A} depend on the linear size ℓ of the subsystem in the different cases?

(3.b) Generalize the situation from (2.a) to three dimensions, and a cubic subsystem of size $\ell\times\ell\times\ell$. How does the entanglement entropy S_{A} depend on the linear size ℓ of the subsystem in this case?

(3.c) Now consider a 2D parent Hamiltonian in which every spin $\hat{\boldsymbol{S}}_{\boldsymbol{i}}$ is coupled to *exactly one* other spin $\hat{\boldsymbol{S}}_{\boldsymbol{j}}$ by an antiferromagnetic Heisenberg coupling; denote by $(\boldsymbol{i}, \boldsymbol{j})$ a pair of coupled spins, such that the parent Hamiltonian becomes:

$$\hat{\mathcal{H}} = J\sum_{(\boldsymbol{i},\boldsymbol{j})} \hat{\boldsymbol{S}}_{\boldsymbol{i}} \cdot \hat{\boldsymbol{S}}_{\boldsymbol{j}} \tag{12}$$

(i) Assume that no couplings exist between spins at sites $\boldsymbol{i}, \boldsymbol{j}$ further apart than a critical distance d_c. Otherwise, assume that the combinations of coupled spins is random. For this situation, how does the entanglement entropy S_A of a rectangular subsystem A of size $\ell \times \ell$ scale with the linear size ℓ, assuming that $\ell \gg d_c$?

(ii) Next, consider a completely random combination of couplings at arbitrary distances, i.e. $d_c = \infty$. How does the entanglement entropy S_A of a rectangular subsystem A scale with its linear size ℓ in this case?

(3.d) Consider a system of N distinguishable and non-interacting quantum particles in a box of length L. Assume that each particle is prepared in the quantum mechanical ground state $|\Phi_0\rangle$ of the box, i.e.:

$$|\Psi\rangle = \prod_{n=1}^{N} |\Phi_0\rangle_n. \tag{13}$$

Divide the system into two spatial parts A and B defined by its left and right halfs, respectively. Calculate the entanglement entropy S_A – how does it depend on the particle number N?