Prof. Dr. Stefan Hofmann Winter term 2019/20

Exercises on General Relativity TVI TMP-TC1

Problem set 5

Exercise 1 – EMT Self-Coupling

The Lagrange density of a free massless scalar field is

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi\,. \tag{1}$$

Take this theory and introduce self-coupling by adding the interaction term

$$\mathcal{L}_I = g\phi\,\mathrm{tr}_\eta(T), \tag{2}$$

where g is a dimensionless coupling constant and T the energy-momentum tensor (EMT) of $\mathcal{L}$. Derive the new energy momentum tensor T_I of the theory $\mathcal{L} + \mathcal{L}_I$. Now repeat this procedure defining the second interaction term as

$$\mathcal{L}_{II} = g\phi\,\mathrm{tr}_\eta(T_I)\,. \tag{3}$$

Discuss the conditions that need to hold for the coupling constant g for such an iteration to be feasable.

Exercise 2 – The Fierz-Pauli action functional

Consider the Fierz-Pauli action introduced in the lecture expressed in inertial coordinates:

$$S[h] = -\frac{1}{2G'_N}\int_{M_4} h^{\alpha\beta}\Box\left(P_\alpha{}^\mu P_\beta{}^\nu - P_{\alpha\beta}P^{\mu\nu}\right)h_{\mu\nu} \tag{4}$$

where P is the perpendicular projector $P_\perp$.
Show that the above functional is invariant under $\delta_\epsilon h_{\mu\nu} = \partial_{(\mu}\epsilon_{\nu)}$ with ϵ a smooth vector field.

Exercise 3 – Coupling of the spin-2 field to a massive scalar field

(i) Use the Fierz-Pauli action $S^k[h]$ from equation (4) and vary it to derive the equations of motion, namely

$$E_{\mu\nu} := \Box h_{\mu\nu} + \partial_\mu \partial_\nu h - \partial^\alpha \partial_{(\mu} h_{\nu)\alpha} - \eta_{\mu\nu} \left(\Box h - \partial_\alpha \partial_\beta h^{\alpha\beta} \right) = 0 \,, \tag{5}$$

where we used the shorthand notation $h := h_\mu{}^\mu$.

(ii) In order to couple this field to a massive scalar, we introduce a source term $S_I[h]$ to the Fierz-Pauli action,

$$S_I[h] = \int_{M_4} h_{\mu\nu} T^{\mu\nu} \,, \tag{6}$$

where T is the energy-momentum tensor (EMT), changing the equations of motion to $E'_{\mu\nu} = E_{\mu\nu} + E^I_{\mu\nu} = 0$. Write down the new expression for the equation of motion explicitly.

(iii) We now consider the example of a scalar scource field (at rest) with mass $M \in \mathbb{R}^+$. The corresponding EMT is given by

$$T^{\mu\nu}(\boldsymbol{x}) = M \eta_0^\mu \eta_0^\nu \delta^{(3)}(\boldsymbol{x}) \,. \tag{7}$$

Argue, why this is a reasonable EMT for the system we want to describe.

(iv) Introduce for convenience the variable $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$ and eliminate the non-physical degrees of freedom of the massless spin-2 field using harmonic gauge $\partial^\mu \bar{h}_{\mu\nu} = 0$ at the level of the equations of motion.

(v) Count the remaining (physical) degrees of freedom that propagate, under the assumption that $h_{\mu\nu}$ is a totally symmetric tensor and solve the equations of motion $E'_{\mu\nu} = 0$ for $\bar{h}_{\mu\nu}$.

(vi) Finally, express the components of the field, h_{00} and h_{ij}, using the definition $\Phi(\boldsymbol{x}) = -\frac{\bar{G}_N M}{4\pi\,|\boldsymbol{x}|}$ and give a physical interpretation for h_{00}.

General information

The lecture takes place on Monday at 10:00-12:00 and on Wednesday at 10:00-12:00 in A348.

Presentation of solutions:
Thursday at 08:00 - 10:00 in A 348

There are two tutorials:
Monday at 12:00 - 14:00 in A 249
Friday at 14:00 - 16:00 in A 348

The webpage for the lecture and exercises can be found at

`www.physik.uni-muenchen.de/lehre/vorlesungen/wise_19_20/tvi_tc1_gr/index.html`