Prof. Dr. Stefan Hofmann Winter term 2019/20

Exercises on General Relativity TVI TMP-TC1

Problem set 4

Exercise 1 – General Noether's theorem

Consider the following Lagrangian density $\mathcal{L}(\phi^i, \partial_\mu \phi^i)$ in a D dimensional Minkowski space-time $x^\mu \in \mathbb{M}_D$ and ϕ^i are N scalar fields.
We assume that the following transformations

$$\tilde{x}^\mu = x^\mu + \delta x^\mu$$
$$\tilde{\phi}^i(\tilde{x}) = \phi^i(x) + \delta\phi^i(x)$$

leave the action functional invariant i.e

$$S = \int_\Omega \mathcal{L}(\phi^i(x), \partial_\mu \phi^i(x))\ \mathrm{d}^D x = \int_\Omega \mathcal{L}(\tilde{\phi}^i(\tilde{x}), \tilde{\partial}_\mu \tilde{\phi}^i(\tilde{x}))\ \mathrm{d}^D \tilde{x} \tag{1}$$

Show that if ϕ is a solution of the Euler-Lagrange equations then there exists a conserved current J^μ in the sense that $\partial_\mu J^\mu = 0$

Hint: You can assume that $\int_\Omega \delta\mathcal{L}\ \mathrm{d}^D x = 0\ \Rightarrow \delta\mathcal{L} = \partial_\mu F^\mu$ *for some field* F^μ *that vanishes on* $\partial\Omega$.

The solution should have the following form:

$$J^\mu = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)}\delta\phi^i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)}\partial_\nu \phi^i \delta x^\nu - \mathcal{L}\,\delta x^\mu + F^\mu \tag{2}$$

Exercise 2 – Energy-momentum tensor of the free scalar field

Given the following Lagrange density of a free massive scalar field with an external scource $J(\boldsymbol{x})$:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + J\phi \tag{3}$$

(i) Determine the equations of motion for the field ϕ and analyze whether this equation transforms invariantly under Galileo and Lorentz transformations.

(ii) Use the condition that $\mathcal{L}$ does not depend explicitly on the space-time coordinates (it is invariant under space-time translations) to derive the energy momentum tensor $T^{\mu\nu}$. For $J = 0$ show that it is conserved, i.e. $\partial_\mu T^{\mu\nu} = 0$.

(iii) Prove that $T^{00} \geq 0$ and give its physical interpretation.

Exercise 3 – Conserved current in classical electrodynamics

The Lagrangian of classical electrodynamics (massless vector field) is the following:

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} \,, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{4}$$

Use the condition that $\mathcal{L}$ does not depend explicitly on the space-time coordinates (it is invariant under space-time translations) to derive the following equation

$$\partial_\mu \theta^{\mu\nu} = 0$$

where $\theta^{\mu\nu}$ is the corresponding Noether current.
In order to make $\theta^{\mu\nu}$ symmetric and gauge invariant add a suitable conserved quantity. Denote the new quantity as $T^{\mu\nu}$ and write it down under the assumption that the positive definiteness of T^{00} should be fulfilled. Spell out the component T^{00} in terms of the electric and magnetic field.
Take $T^{\mu\nu}$ and check whether a vertex density v, which couples electromagnetic fields to a scalar gravity field Φ, is possible as discussed in the lecture:

$$v = \Phi \, \mathrm{tr}_\eta(T) \,. \tag{5}$$

Recall why this rules out the scalar model for gravity.

General information

The lecture takes place on Monday at 10:00-12:00 and on Wednesday at 10:00-12:00 in A348.

Presentation of solutions:
Thursday at 08:00 - 10:00 in A 348

There are two tutorials:
Monday at 12:00 - 14:00 in A 249
Friday at 14:00 - 16:00 in A 348

The webpage for the lecture and exercises can be found at

`www.physik.uni-muenchen.de/lehre/vorlesungen/wise_19_20/tvi_tc1_gr/index.html`