Prof. Dr. Stefan Hofmann Winter term 2019/20

Exercises on General Relativity TVI TMP-TC1

Problem set 2

Exercise 1 – “Paradoxes”

(i) **Ladder Paradox** A garage with two doors on opposite sides is at rest in Bob’s frame of reference. Furthermore, let there be a ladder longer than the distance between the two doors in Bob’s inertial system. Alice holds the ladder horizontally and runs with velocity close to c in the direction of the two doors. In Bob’s frame the length of the ladder is contracted in such a way, that it now can fit in the garage. From Alice’s perspective, however, the garage is shortened and the ladder does not fit in it.

Can Bob close the doors in a way such that the ladder is in the garage? Can Alice do the same? If not, is this a contradiction?

(ii) **Twin Paradox** Consider the two twins Alice and Bob. Bob travels with a spaceship 3 light years (as seen by Alice) away, at a speed $v = 0.6$. After arriving at the turning point, he travels back to Earth where Alice waited for him. Since they cannot measure their biological age, calculate their age difference due to time dilation. Furthermore, explain the effect using Minkowski diagrams, note that they exchanged light signals each time their watch told them that 1 year had passed.

(iii) **Cuboid** Consider a cuboid traveling with velocity v in a direction perpendicular to an observer looking at it. The light rays reaching the observer are assumed to be parallel to each other, which is a valid approximation if the observer is sufficiently far away. See Figure 1 for a schematic representation of the system.

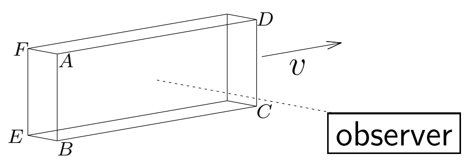

Figure 1:

Light reaching the observer from the back of the cuboid, e.g. from the points E and F, takes longer to reach the observer than light coming from the front surface, delimited by the points A, B, C, and D.

Show that taking this effect and the length contraction into account the observer sees a rotated cuboid. Furthermore determine the angle of its rotation as a function of v.

What would the observer see, if the cuboid were to be replaced by a sphere?

Exercise 2 – Principle of relativity

The goal of this exercise is to derive the Lorentz transformations using the following assumptions. This approach is similar to the one Einstein used to derive them in the paper "On the Electrodynamics of Moving Bodies" from 1905.

- The space is isotropic, i.e. all space directions are equivalent.
- Space and time are homogeneous, i.e. all points are equivalent and the choice of the origin of the coordinate system is arbitrary.
- The relativity principle holds, i.e. the laws of physics are unchanged in all inertial systems, thus for all observers which are at rest or moving with a constant velocity.

In order to derive the Lorentz transformations proceed as follows.

(i) Consider two inertial frames K' and K. K' is moving with velocity v along the x direction with respect to K and has coordinates $\{t', x', y', z'\}$, K has coordinates $\{t, x, y, z\}$. Furthermore assume that the origins of the two coincide at $t = t' = 0$. This means that after a time t the origin of K' will have coordinates $\{x = vt, y = 0, z = 0\}$ in the K coordinate system.

The generic transformation between the two systems must then take the form

$$\begin{aligned} x' &= \gamma(v)(x - vt) \\ y' &= \alpha(v)\, y \\ z' &= \alpha(v)\, z \\ t' &= \mu(v)\, t + \varepsilon(v)\, x \ , \end{aligned} \tag{1}$$

where γ, α, μ, and ε are some functions depending only on v which will be determined later. Motivate Eq. (1), by following these steps.

a) Why must the transformations be linear?

b) Why can γ, α, μ, and ε only depend on v?

c) Why does the same function $\alpha(v)$ appear both for y' and for z'?

d) Why do only x and t appear in t', but not y and z?

(ii) Since no direction is preferred, the transformations should not change if we invert the orientation of all axes and take $-v$ instead of $+v$ for the velocity. Use this to determine whether the functions γ, α, μ, and ε are even or odd.

Furthermore, eliminate $\varepsilon(v)$ in Eq. (1) by defining a new function

$$\eta(v) = -\frac{v}{\varepsilon(v)}\mu(v) \ . \tag{2}$$

Is η even or odd?

(iii) By the relativity principle the transformation $K' \to K$ should be equivalent to the transformation $K \to K'$ with opposite sign for v. Use this to show that $\alpha^2(v) = 1$ and motivate why $\alpha(v) = +1$ should be taken.

Furthermore find an expression for γ and μ as functions of η.

(iv) Finally use the relativity principle again, by saying that $K \to K' \to K''$ should have the same form as Eq. (1), to show that η must be a constant.

As far as we know the maximal velocity $\sqrt{\eta}$ is the speed of light in vacuum. If we plug this into Eq. (1) we find the usual Lorentz transformations.

Exercise 3 – Energy-Mass equivalence and the Einstein Mirror

(i) In this exercise we go through the paper "On the Electrodynamics of Moving Bodies" and "Does the Inertia of a Body Depend upon its Energy Content?" published in 1905 in order to understand how Einstein found the equation $E = mc^2$.

a) Imagine a sphere which travels with the speed of light in the reference system we call lab frame. Show that the volume of the sphere in a reference system boosted with velocity v is given by

$$\frac{V'}{V} = \sqrt{\frac{1+v}{1-v}}\,. \tag{3}$$

b) We can arrange the system such that a plane wave enclosed in this sphere is enclosed forever and the energy content within the sphere does not change with time. The energy within the sphere measured in the boosted reference system is

$$\frac{E'}{E} = \frac{1-v}{\sqrt{1-v^2}}\,. \tag{4}$$

Now let us perform the following Gedankenexperiment: A body with energy E_0 and mass m is at rest w.r.t. to the lab frame. It sends out two electromagnetic waves in opposite directions each with energy $\frac{L}{2}$. The body has a remaining energy of E_1. In the boosted reference frame the body has the initial energy H_0 and final energy H_1. With the formula above we know how the energy of the electromagnetic waves changes under a Lorentz boost. Use this to derive that to radiate away the energy L the body had to reduce its mass by $\frac{L}{c^2}$. Therefore, the following equation holds:

$$E = mc^2 \tag{5}$$

(ii) Now let us consider the first paper again. In it what will be later called an Einstein mirror is discussed. It consists of a relativistically moving mirror ($v \approx 0.1$) on which light is reflected perpendicular to its surface. First of all, show that the frequency in a boosted reference system is given by

$$\frac{\nu'}{\nu} = \sqrt{\frac{1-v}{1+v}} \tag{6}$$

Derive the frequency of the reflected wave w.r.t. the initial frequency in the lab frame and discuss the limits $v \to 1$ and $v \to 0$.

General information

The lecture takes place on Monday at 10:00-12:00 and on Wednesday at 10:00-12:00 in A348.

Presentation of solutions:
Thursday at 08:00 - 10:00 in A 348

There are two tutorials:
Monday at 12:00 - 14:00 in A 249
Friday at 14:00 - 16:00 in A 348

The webpage for the lecture and exercises can be found at

`www.physik.uni-muenchen.de/lehre/vorlesungen/wise_19_20/tvi_tc1_gr/index.html`