Exercises on General Relativity TVI TMP-TC1

Problem set 12

Exercise 1 – Vectors, 1-forms, and the metric tensor

(i) A symmetric, non-degenerate bilinear form on the tangent space of a manifold is called a metric. For a given metric g and a vector u, we can define a 1-form ω_u by its action on an arbitrary vector v:

$$\omega_u(v) = g(u, v) \; . \tag{1}$$

Consider a coordinate basis of vectors $\frac{\partial}{\partial x^\mu}$ and 1-forms $\mathrm{d}x^\mu$. Show that the components of the 1-form ω_u are given by the formula

$$(\omega_u)_\mu = g_{\mu\nu} u^\nu \; , \tag{2}$$

where $g_{\mu\nu}$ are the components of the metric in a given coordinate system.

(ii) Consider the 2-sphere $N = \mathbb{S}^2$ embedded in $M = \mathbb{R}^3$. The coordinates on N and M are denoted as $x^\mu = (\theta, \phi)$ and $y^\mu = (x, y, z)$ respectively for $\theta \in [0, \pi)$ and $\phi \in [0, 2\pi)$. Then the map $\Phi : N \to M$ is given by

$$\Phi(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \; . \tag{3}$$

Find the metric on $\mathbb{S}^2$ by pulling back the Euclidean metric from $\mathbb{R}^3$.

(iii) Now take the vector field defined on the surface N as

$$V = \cos\phi\,\partial_\phi + \sin\phi\,\partial_\theta \tag{4}$$

and check whether it is a unit vector and compute its components in M.

Exercise 2 – Christoffel symbols

(i) Consider the Levi-Civita connection and find the transformation property of the Christoffel symbols Γ under arbitrary coordinate transformations:

$$\tilde{\Gamma}^\alpha_{\beta\gamma} = \Gamma^\mu_{\rho\sigma} \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial x^\rho}{\partial \tilde{x}^\beta} \frac{\partial x^\sigma}{\partial \tilde{x}^\gamma} + \frac{\partial^2 x^\sigma}{\partial \tilde{x}^\beta \partial \tilde{x}^\gamma} \frac{\partial \tilde{x}^\alpha}{\partial x^\sigma} \; , \tag{5}$$

where Γ is the Christoffel symbol in coordinates x and $\tilde{\Gamma}$ in coordinates $\tilde{x}$. Find a subset of the general coordinate transformations under which the Christoffel symbol transforms like a tensor. What does this tell you about the connection in arbitrary Cartesian coordinates if all Christoffel symbols vanish in a given Cartesian coordinate system?

(ii) The non-vanishing Christoffel symbols of $\mathbb{S}^2$ in the coordinates $x^\mu = (\theta, \phi)$ of problem 12.1 have already been computed in Problem Sheet 8, Exercise 3 (in that case for a generic radius R). Compute them once more using Mathematica. You can find Mathematica Notebooks with expressions for calculating the Christoffel symbols and much more at `http://web.physics.ucsb.edu/~gravitybook/mathematica.html`.

(iii) Show that in these coordinates the Christoffel symbols fail to describe geodesics going through $\theta = \{0, \pi\}$. Find coordinates in which you can circumvent this problem. Although one obviously cannot cover $\mathbb{S}^2$ globally by Euclidean coordinates, can you think of a way how this can be introduced in the neighborhood of an arbitrary point on the sphere?

(iv) Now consider two sets of connection coefficients Γ and $\hat{\Gamma}$. Take their difference

$$\Gamma^{\alpha}_{\beta\gamma} - \hat{\Gamma}^{\alpha}_{\beta\gamma} \tag{6}$$

and find whether it transforms as a tensor under arbitrary coordinate transformations. What are the consequences?

Exercise 3 – Bending of light

(i) Analogously to problem 11.3 for a massive particle we find the equation of motion for massless particles in the (r, ϕ) plane to be

$$\dot{r}^2 = E^2 - \frac{L^2}{r^2} + \frac{2ML^2}{r^3} \qquad \dot{\phi} = \frac{L}{r^2} \tag{7}$$

Use these equations to obtain the following differential equation for the orbit:

$$u'' + u = 3Mu^2 \tag{8}$$

where $u = \frac{1}{r}$ and $u' = \frac{\mathrm{d}u}{\mathrm{d}\phi}$ by differentiating $\frac{\dot{r}^2}{\dot{\phi}^2}$ once.

(ii) For $M = 0$ equation (8) is solved by $u = \frac{\sin\phi}{R_0}$ with the smallest radius of light R_0. Consider a small perturbation around this solution for the case $M \neq 0$ and solve (8). Verify that the gravitational bending of light passing near the sun is

$$\Delta\phi = 1.75'' \frac{R_\odot}{R_0} \tag{9}$$

where $R_\odot$ is the radius of the sun.

General information

The lecture takes place on Monday at 10:00-12:00 and on Wednesday at 10:00-12:00 in A348.

Presentation of solutions:
Thursday at 08:00 - 10:00 in A 348

There are two tutorials:
Monday at 12:00 - 14:00 in A 249
Friday at 14:00 - 16:00 in A 348

The webpage for the lecture and exercises can be found at

`www.physik.uni-muenchen.de/lehre/vorlesungen/wise_19_20/tvi_tc1_gr/index.html`