Prof. Dr. Stefan Hofmann Winter term 2019/20

Exercises on General Relativity TVI TMP-TC1

Problem set 11

Exercise 1 – Pullback of the field-strength tensor

Consider the field-strength tensor of electrodynamics in cartesian coordinates, i.e.

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} . \tag{1}$$

Express it in spherical coordinates by performing a pullback.

Compute both $F_{\mu\nu}F^{\mu\nu}$ and $F_{AB}F^{AB}$, where capital latin letters indicate that the tensor is given in spherical coordinates.

Exercise 2 – Symmetries and Killing vector fields

A metric has a symmetry if there is a coordinate transformation that does not change the components of the metric. State all symmetry transformations of the Minkowski metric which in Cartesian coordinates is given by diag$(-,+,+,+)$.
An arbitrary infinitesimal coordinate transformation can be written as

$$\tilde{x}^\alpha = x^\alpha + \varepsilon\xi^\alpha \tag{2}$$

with $\varepsilon \ll 1$. This coordinate transformation is a symmetry of the metric if the vector field ξ satisfies the Killing equation which is in Minkowski space-time given by

$$\partial_\alpha\xi_\beta + \partial_\beta\xi_\alpha = 0 \tag{3}$$

(i) How many independent equations are included in (3) and how many parameters have to be fixed in order to address all symmetry transformations of Minkowski?

(ii) For the spatial translation $\tilde{x}^1 = x^1 + \varepsilon\xi^1$ the vector field is $\xi^\alpha = (0,1,0,0)$. Show that it solves (3).

(iii) Find the Killing vector fields corresponding to all remaining symmetry transformations for the Minkowski metric.

Exercise 3 – Geodesics in the space-time of the sun

Approximate the space-time outside of the sun with mass M as

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -f(r)\,\mathrm{d}t^2 + \frac{1}{f(r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 \tag{4}$$

with $f(r) := \left(1 - \frac{2GM}{r}\right)$ and $\mathrm{d}\Omega = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$.

(i) Convince yourself that the limit $r \to \infty$ is correct.

(ii) Write down the Lagrangian of a massive point particle moving in this space-time.

(iii) Exploit the symmetries of the metric to find the corresponding constants of motion E and L.

From now on restrict to the equatorial plane $\theta = \frac{\pi}{2}$.

(iv) Find a relation for E^2 using $u^\mu u_\mu = -1$ with the four velocity u.

(v) Subtracting the rest energy and the radial kinetic energy, find the effective potential for this particle.

(vi) Sketch the effective potential for $L \neq 0$, discuss possible geodesics and take the non-relativistic limit. From these results how can you decide whether you have to use General Relativity or Newton's gravity to describe the solar system?

General information

The lecture takes place on Monday at 10:00-12:00 and on Wednesday at 10:00-12:00 in A348.

Presentation of solutions:
Thursday at 08:00 - 10:00 in A 348

There are two tutorials:
Monday at 12:00 - 14:00 in A 249
Friday at 14:00 - 16:00 in A 348

The webpage for the lecture and exercises can be found at

`www.physik.uni-muenchen.de/lehre/vorlesungen/wise_19_20/tvi_tc1_gr/index.html`