Exercises on General Relativity TVI TMP-TC1

Sample solution to problem set 4 of PS 13

Solution

(i) The corresponding metric is given by

$$g_{\mu\nu} = \mathrm{diag}\left(-1, \frac{a^2}{1-kt^2}, a^2r^2, a^2r^2\sin^2\theta\right) \tag{1}$$

$$g^{\mu\nu} = \mathrm{diag}\left(-1, \frac{1-kr^2}{a^2}, \frac{1}{a^2r^2}, \frac{1}{a^2r^2\sin^2\theta}\right) . \tag{2}$$

Christoffel symbols; with help of an algebraic program we compute all non-vanishing components:

$$\Gamma^t_{rr} = \frac{\dot{a}a}{1-kr^2}\,, \quad \Gamma^t_{\theta\theta} = r^2 a\dot{a}\,, \quad \Gamma^t_{\phi\phi} = r^2 a\dot{a}\sin^2\theta \tag{3}$$

$$\Gamma^r_{tr} = \frac{\dot{a}}{a}\,, \quad \Gamma^r_{rr} = \frac{kr}{1-kr^2}\,, \quad \Gamma^r_{\theta\theta} = -r(1-kr^2) \tag{4}$$

$$\Gamma^r_{\phi\phi} = -r(1-kr^2)\sin^2\theta\,, \quad \Gamma^\theta_{r\theta} = \frac{\dot{a}}{a}\,, \quad \Gamma^\theta_{r\theta} = \frac{1}{r} \tag{5}$$

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta\,, \quad \Gamma^\phi_{t\phi} = \frac{\dot{a}}{a}\,, \quad \Gamma^\phi_{r\phi} = \frac{1}{r}\,, \quad \Gamma^\theta_{\phi\theta} = \frac{\cos\theta}{\sin\theta}\,. \tag{6}$$

Riemann tensor:

$$R^t{}_{rtr} = \frac{\ddot{a}a}{1-kr^2}\,, \quad R^t{}_{\theta t\theta} = r^2 a\ddot{a}\,, \quad R^t{}_{\phi t\phi} = r^2 a\ddot{a}\sin^2\theta \tag{7}$$

$$R^r{}_{ttr} = \frac{\ddot{a}}{a}\,, \quad R^r{}_{\theta r\theta} = r^2(k+\dot{a}^2)\,, \quad R^r{}_{\phi r\phi} = r^2\sin^2\theta(k+\dot{a}^2) \tag{8}$$

$$R^\theta{}_{tt\theta} = \frac{\ddot{a}}{a}\,, \quad R^\theta{}_{rr\theta} = -\frac{k+\dot{a}^2}{1-kr^2}\,, \quad R^\theta{}_{\phi\theta\phi} = r^2\sin^2\theta(k+\dot{a}^2) \tag{9}$$

$$R^\phi{}_{tt\phi} = \frac{\ddot{a}}{a}\,, \quad R^\phi{}_{rr\phi} = -\frac{k+\dot{a}^2}{1-kr^2}\,, \quad R^\phi{}_{\theta\theta\phi} = -r^2(k+\dot{a}^2)\,. \tag{10}$$

Ricci tensor:

$$R_{tt} = \frac{3\ddot{a}}{a}\,, \quad R_{rr} = \frac{2k+2\dot{a}^2+a\ddot{a}}{1-kr^2} \tag{11}$$

$$R_{\theta\theta} = r^2(2k+2\dot{a}^2+a\ddot{a})\,, \quad R_{\phi\phi} = r^2(2k+2\dot{a}^2+a\ddot{a})\sin^2\theta\,. \tag{12}$$

Ricci scalar:

$$R = 3\frac{\ddot{a}}{a} + \left(\frac{2k+2\dot{a}^2+a\ddot{a}}{a^2}\right)\cdot 3 = 6\left(\frac{k}{a}+\frac{\dot{a}^2}{a^2}+\frac{\ddot{a}}{a}\right)\,. \tag{13}$$

(ii) If $k=0$, defining $\mathrm{d}t = a\,\mathrm{d}\eta$ we have

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2) \tag{14}$$

$$= -a^2\mathrm{d}\eta^2 + a^2(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2) \tag{15}$$

$$= a^2(-\mathrm{d}\eta^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2)\,. \tag{16}$$

This metric is proportional to the flat metric, with a proportionality factor depending on the coordinate t. We say that g is conformally flat.

(iii) Let x^μ be the position of some galaxy in the universe. The galaxy is not moving. Let us derive the components of the $T_{\mu\nu}$ in spherical coordinates:

$$T_{00} = (\epsilon + p)(-1)^2 + p(-1) = \epsilon \tag{17}$$

$$T_{rr} = 0 + pg_{rr} = p\frac{a^2}{1-kr^2} \tag{18}$$

$$T_{\theta\theta} = pr^2a^2 \tag{19}$$

$$T_{\phi\phi} = pr^2a^2\sin^2\theta \tag{20}$$

Taking the trace of the Einstein equations and solving for R, we obtain

$$R_{\mu\nu} = 8\pi G_N(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T^\kappa_\kappa) + g_{\mu\nu}\Lambda \tag{21}$$

with

$$T^\mu_\mu = g^{\mu\nu}T_{\mu\nu} = -\epsilon + 3p \tag{22}$$

and for the component 00

$$-\frac{\ddot{a}}{a} = 4\pi G_N(p + \tfrac{1}{3}\epsilon) - \frac{1}{3} \tag{23}$$

and for the component rr

$$\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{k}{a} = -4\pi G_N(p - \epsilon) + 1\,. \tag{24}$$

Using the first equation into the second we derive the Friedmann equations:

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G_N}{3}\epsilon + \frac{1}{3}\Lambda \tag{25}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3}(\epsilon + 3p) + \frac{1}{3}\Lambda \tag{26}$$