Black Holes and their Thermodynamics[1]

Problem Sheet 9

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Tutorials take place on Mondays, 2-4pm (c.t.) in room A 449, Theresienstr. 37. Please hand in your solutions at the next tutorial on December 16, 2019 or send them in a single pdf in an email before the tutorial.

Exercise 1 *Take the Kerr metric and show that the two roots $r_+ > r_-$ of the function Δ are Killing horizons of the Killing vector fields $\xi_\pm = \partial_t + \Omega_\pm \partial_\phi$, where $\Omega_\pm$ are constants that you should determine. One interpretation of this result is that the event horizon (i.e., the outer Killing horizon $r = r_+$) of the Kerr black hole rotates with angular velocity Ω_+. One can prove that on the event horizon, the surface gravity is given by*

$$\kappa_+ = \frac{r_+ - r_-}{2(r^2 + a^2)} \,. \tag{1}$$

Derive a condition for vanishing κ_+ in terms of M and a.

Exercise 2 *In the ergoregion of a Kerr black hole, the Killing vector $\xi = \partial_t$ is spacelike. Therefore, particles with 4-momentum p can have negative energy in the ergoregion, $E = -p_\mu \xi^\mu < 0$. Forcing the black hole to absorb such a particle with negative energy means that we can extract positive energy from the black hole. This process is called the Penrose process. From the causal inequality at the outer Killing horizon $p_\mu \xi_+^\mu \leq 0$(where $\xi_\pm = \partial_t + \Omega_\pm \partial_\phi$ from the previous exercise), derive an inequality that relates energy $E < 0$ and angular momentum L of the infalling particle. Assume that the final black hole state has mass $M + \delta M$ and angular momentum $J + \delta J$ where $\delta M = E$ and $\delta J = L$. Derive now the inequality*

$$\delta(M^2 + \sqrt{M^4 - J^2}) \geq 0 \,. \tag{2}$$

Show also that the result above can be rewritten as

$$\delta A \geq 0 \tag{3}$$

with $A = 4\pi(r_+^2 + a^2)$. This is a particular example of the area theorem, which states that the surface area of a black hole can never decrease.

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[1] $www.physik.uni-muenchen.de/lehre/vorlesungen/wise\_19\_20/bh\_info\_wise\_2019\_20$

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Exercise 3 *For every Killing vector ξ, one can define a Komar current*

$$J^\mu_{(\xi)} = R^{\mu\nu}\xi_\nu\,. \tag{4}$$

1. *Show that*

$$J^\mu_{(\xi)} = \nabla_\nu\nabla^\mu\xi^\nu \tag{5}$$

*(**Hint:** The relation $\nabla_\mu\nabla_\nu\xi^\alpha = R^\alpha{}_{\nu\mu\beta}\xi^\beta$ that you proofed on an earlier problem sheet might be useful.)*

2. *Use this to show*

$$\nabla_\mu J^\mu_{(\xi)} = 0\,. \tag{6}$$

*(**Hint:** $[\nabla_\mu\nabla_\nu]T^{\rho\sigma} = R^\rho_{\alpha\mu\nu}T^{\alpha\sigma} + R^\rho_{\alpha\mu\nu}T^{\rho\alpha}$ for any $(2,0)$-tensor $T^{\rho\sigma}$.)*

Equation (6) implies the conservation of the charge passing through a space-like hypersurface Σ, i.e.

$$Q_{(\xi)} \sim \int_\Sigma \mathrm{d}^3x\sqrt{\gamma}n_\mu J^\mu_{(\xi)} \tag{7}$$

where γ_{ij} is the induced metric on Σ and n_μ is the unit future-pointing normal vector of Σ. On the other hand, equation (5) implies that the charge can be written as a surface integral

$$Q_{(\xi)} \sim \int_{\partial\Sigma} \mathrm{d}^2x\sqrt{\gamma^{(2)}}n_\mu\sigma_\nu\nabla^\mu\xi^\nu\,, \tag{8}$$

where the boundary $\partial\Sigma$, typically a two-sphere at spatial infinity, has metric $\gamma^{(2)}_{ij}$ and outward pointing unit normal vector σ, n . For $\xi = \partial_t = k$ and for appropriate choice of the overall constant this gives the total energy of a stationary spacetime, whereas for $\xi = \partial_\phi = m$ it leads to its conserved angular momentum, i.e.

$$E = \frac{1}{4\pi G}\int_{\partial\Sigma} \mathrm{d}^2x\sqrt{\gamma^{(2)}}n_\mu\sigma_\nu\nabla^\mu k^\nu \tag{9}$$

$$J = \frac{-1}{8\pi G}\int_{\partial\Sigma} \mathrm{d}^2x\sqrt{\gamma^{(2)}}n_\mu\sigma_\nu\nabla^\mu m^\nu\,. \tag{10}$$

3. *Extra: Determine the vectors n_μ and σ_μ (normalized to $n^2 = -1$ and $\sigma^2 = 1$) which are normal to $t = \text{constant}$ and $r = \text{constant}$ surfaces*

for the Kerr metric. Then evaluate the Komar integrals for $\partial\Sigma$ the two-sphere at infinity with induced metric

$$\gamma^{(2)}_{ij}\mathrm{d}x^i\mathrm{d}x^j = r^2\mathrm{d}\Omega^2 \tag{11}$$

*(**Hint:** You can use a computer algebra program to compute the components of the inverse metric and the Christoffel symbols that you need. You can use, e.g., the mathematica notebook "Curvature and the Einstein Equation" that you can find on http://web.physics.ucsb.edu/ gravitybook/mathematica.html)*