Black Holes and their Thermodynamics[1]

Problem Sheet 8

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Tutorials take place on Mondays, 2-4pm (c.t.) in room A 449, Theresienstr. 37. Please hand in your solutions at the next tutorial on December 9, 2019 or send them in a single pdf in an email before the tutorial.

Exercise 1 *When a black hole spacetime admits a Killing vector ξ, it may occur that the black hole horizon $\mathcal{H}$ is also a Killing horizon. (A Killing horizon is defined as a null hypersurface on which $\xi_a\xi^a = 0$.) On such horizons, the surface gravity κ is defined through*

$$\xi^\sigma \nabla_\sigma \xi^\mu \Big|_{\mathcal{H}} = \kappa \xi^\mu \,. \tag{1}$$

One can show that

$$\kappa^2 = -\frac{1}{2}(\nabla_\mu \xi_\nu)(\nabla^\mu \xi^\nu)\Big|_{\mathcal{H}} \,. \tag{2}$$

1. *Take the metric*

$$\mathrm{d}s^2 = -A(r)\mathrm{d}t^2 + \frac{1}{A(r)}\mathrm{d}r + r^2 \mathrm{d}\Omega^2 \tag{3}$$

and assume that $A(r)$ has at least one root at $r = r_0$. Argue that ∂_t is a Killing vector, and show that the metric has a Killing horizon at $r = r_0$. Then show that its associated surface gravity is

$$\kappa = \frac{1}{2}\frac{\mathrm{d}A}{\mathrm{d}r}\Big|_{r=r_0} \,. \tag{4}$$

2. *Compute κ for the Schwarzschild spacetime.*

3. *When does the surface gravity vanish?*

4. *Let $x(\tau)$ be an integral curve of of the Killing vector ξ. Show that*

$$a^\mu = \kappa \dot{x}^\mu \,, \tag{5}$$

where a^μ is the acceleration of the curve.

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Exercise 2 *The Reissner-Nordström metric in ingoing Eddington-Finkelstein coordinates reads*

$$\mathrm{d}s^2 = -\Delta \mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2\mathrm{d}\Omega^2\,, \tag{6}$$

where $\Delta = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$. *For* $M^2 > Q^2$ *the metric has two event horizons at the zeros of* Δ, *i.e. at* $r_\pm = M \pm \sqrt{M^2 - Q^2}$.

Show that the outer and inner horizons located at $r = r_\pm$ *are null hypersurfaces. Show that they are Killing horizons with respect to the stationary Killing vector* $\xi = \partial_v$. *By calculating* $\nabla_\mu(\xi^2)|_{r=r_\pm}$ *show that the surface gravity on the two Killing horizons is given by*

$$\kappa = \frac{r_\pm - r_\mp}{2r_\pm^2}\,. \tag{7}$$

Exercise 3 *In the presence of an electromagnetic field, a particle of charge* q *and mass* m *obeys the equation of motion*

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = \frac{q}{m}F^\mu{}_\nu\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \tag{8}$$

where τ *is the proper time of the particle. Consider now a charged particle moving in the fields of a Reissner-Nordström black hole with electric charge* Q *and mass* M. *Due to spherical symmetry, orbits of charged particles in the Reissner-Nordström geometry are planar as in the Schwarzschild geometry. We will consider motion in the equatorial plane in the following, i.e.* $\theta = \pi/2$.

1. *Show that* $L = r^2\mathrm{d}\phi/\mathrm{d}\tau$ *is conserved along the motion of the particle.*

2. *Show the conservation of the energy*

$$E = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau} + \frac{q}{m}\frac{Q}{r}\,. \tag{9}$$

3. *Use these conserved quantities to calculate the effective potential* V_{eff} *in*

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + V_{\text{eff}}(r) = \frac{1}{2}(E^2 - 1)\,. \tag{10}$$

4. *Repeat the same calculation for a light ray, i.e. determine the corresponding potential function in*

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 + W_{\text{eff}}(r) = E^2\,, \tag{11}$$

where λ *is an affine parameter along the geodesic (note that the right hand side of eq. (8) vanishes for light rays, so that they follow geodesics).*

5. *Consider now radial plunge trajectories for a neutral particle of unit mass and for a light ray, falling in from infinity. Do they reach the singularity at $r = 0$?*