Black Holes and their Thermodynamics[1]

Problem Sheet 6

Prof. Dieter Lüst[2] and Marvin Lüben[3]

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Tutorials take place on Mondays, 2-4pm (c.t.) in room A 449, Theresienstr. 37. Please hand in your solutions at the next tutorial on November 25, 2019 or send them in a single pdf in an email before the tutorial.

Exercise 1 *(Killing vectors)*

Let $\{\phi_\epsilon, \epsilon \in \mathcal{R}\}$ be a continous family of diffeomorphisms generated by a vector field ξ. The change of the metric components under the action of these diffeomorphisms is given by

$$\phi_\epsilon^* g_{ab} = g_{ab} + \epsilon \mathcal{L}_\xi g_{ab} + \mathcal{O}(\epsilon^2)\,, \tag{1}$$

where $\epsilon \ll 1$ and $\mathcal{L}_\xi$ is called the "Lie derivative",

$$\mathcal{L}_\xi \equiv \frac{\mathrm{d}}{\mathrm{d}\epsilon}(\phi_\epsilon^* g_{ab})\Big|_{\epsilon=0}\,. \tag{2}$$

ξ is called a Killing vector (a symmetry of the metric; isometry) when $\mathcal{L}_\xi g_{ab} = 0$.

1. *From the identity*

$$\mathcal{L}_\xi g_{ab} = \nabla_a \xi_b + \nabla_b \xi_a \tag{3}$$

show that

$$\mathcal{L}_\xi g_{ab} = \xi^c \partial_c g_{ab} + g_{ac}\partial_b \xi^c + g_{bc}\partial_a \xi^c \tag{4}$$

2. *Show that a Killing vector ξ satisfies the equation*

$$\nabla_a \nabla_b \xi^c = R^c{}_{abd}\xi^d\,. \tag{5}$$

(Use the identity $R^c{}_{[abd]} = 0$).

3. *Show that if $\nabla_\mu T^{\mu\nu} = 0$ and ξ is a Killing vector, then $\xi_a T^{ab}$ is conserved.*

[1] $www.physik.uni-muenchen.de/lehre/vorlesungen/wise\_19\_20/bh\_info\_wise\_2019\_20$
[2]dieter.luest@lmu.de
[3]mlueben@mpp.mpg.de

4. *Using (1), show that if ξ is a Killing vector then the geodesic equation implies that*

$$\dot{Q} = 0 \tag{6}$$

with

$$Q = \xi^a p_a \, , \; p_a = g_{ab} \dot{x}^b \, , \tag{7}$$

where dot $\dot{}$ denotes derivative w.r.t. the affine parameter of the geodesic x^a.

These results will soon be useful in the lecture.

Exercise 2 *(Killing vectors in Minkowski and Schwarzschild)*
After having introduced the concept of Killing vectors in the previous exercise, now we now look at concrete examples. Solve the Killing equations

$$\nabla_a \xi_b + \nabla_b \xi_a = 0 \tag{8}$$

for a Killing vector field ξ in order to

1. *find the three linearly independent Killing vectors of the two sphere with metric*

$$\mathrm{d}s^2 = r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) \tag{9}$$

2. *find all (linearly independent) Killing vectors of the Minkowski metric.*

After these preliminaries let us turn to Schwarzschild spacetime.

3. *Show that $\xi = \partial_t$ is a Killing vector (for $r > 2M$). What are the other linearly independent Killing vectors?*
4. *For $r > 2M$, express the Killing vector ξ in terms of Kruskal coordinates (T, R, θ, ϕ).*
5. *Calculate the norm squared of ξ in the Kruskal coordinates and check that it agrees with the calculation in Schwarzschild coordinates. Interpret the sign of the norm of ξ. What is the norm of ξ on the horizon?*

Exercise 3 *(Conformal compactification of Minkowski spacetime)*
Consider the derivation of the conformal compactification of Minkowski spacetime in four spacetime dimensions. Show that $i^0, i^\pm$ are all points and that $\mathcal{I}^\pm$ are topologically $\mathbb{R} \times S^2$.

Exercise 4 *(Motion in the Kruskal-Szekeres manifold)*

1. *Can an observer who falls into a spherical black hole receive information about events that take place outside? Is there any region of spacetime outside the black hole that an interior observer cannot eventually see?*

2. *Suppose that a black hole were really described by the maximal Kruskal-Szekeres extension instead of being produced by a collapsing star. Explain why it would not be possible to traverse from one asymptotic region to the other. But could an observer see light from stars on the other side of the extension?*

3. *An observer falls feet first into a Schwarzschild black hole looking down at her feet. Is there ever a moment when (s)he cannot see her feet? For instance, can (s)he see her feet when her head is crossing the horizon? If so, what radius does (s)he see them at? Does (s)he ever see her/his feet hit the singularity at $r = 0$ assuming (s)he remains intact until her head reaches that radius?*

You can analyze these questions using Kruskal and Eddington-Finkelstein diagrams.