Black Holes and their Thermodynamics[1]

Problem Sheet 5

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Tutorials take place on Mondays, 2-4pm (c.t.) in room A 449, Theresienstr. 37. Please hand in your solutions at the next tutorial on November 18, 2019 or send them in a single pdf in an email before the tutorial.

Exercise 1 *In a previous exercise, we introduced the ingoing Eddington-Finkelstein coordinates* (v, r, θ, ϕ) *in terms of which the line element reads*

$$\mathrm{d}s^2 = -(1 - \frac{2GM}{r})\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2\mathrm{d}\Omega^2\,, \tag{1}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2$ *is the metric on* S^2*. The coordinate* v *in terms of the Schwarzschild coordinate* t *is* $v = t + r^*$ *where*

$$r^* = r + 2GM\ln\left|\frac{r}{2GM} - 1\right|, \tag{2}$$

which is sometimes referred to as "Regge-Wheeler radial coordinate" or "tortoise radial coordinate".

1. *In a previous exercise, you showed that the Schwarzschild metric can be extended through the surface* $r = 2GM$ *to a new region* $r < 2GM$ *via ingoing EF coordinates. For* $r < 2GM$*, show that if the metric (1) is transformed to coordinates* (t, r, θ, ϕ)*, then it becomes the Schwarzschild metric but now with* $r < 2GM$*.*

2. *Introduce the coordinate* $u = t - r^*$ *which follows outgoing light rays. Compute the metric in the Eddington-Finkelstein coordinates* (v, u, θ, ϕ)*.*

3. *In order to eliminate the factor in front of* $\mathrm{d}u\mathrm{d}v$*, Kruskal proposed the following transformation (for* $r > 2GM$*)*

$$\tilde{u} = -e^{-u/(4GM)}\,, \quad \tilde{v} = e^{v/(4GM)}\,. \tag{3}$$

Compute the metric in the new coordinates $(\tilde{v}, \tilde{u}, \theta, \phi)$*. Find an analogous coordinate transformation to eq. (3) for* $r < 2GM$ *in order to arrive at the same metric.*

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4. *Finally, compute the metric in the Kruskal-Szekeres coordinates*

$$T = \frac{1}{2}(\tilde{v} + \tilde{u}) \,, \quad R = \frac{1}{2}(\tilde{v} - \tilde{u}) \,. \tag{4}$$

Your result should coincide with the expression given in eq. (33) in the lecture notes.

5. *What can you say about the geometry defined by the coordinates* (T, R)*?*

6. *Collect your results to find the relation between the Kruskal-Szekeres coordinates in terms of the original Schwarzschild coordinates. Find the (implicit) expressions that define the Schwarzschild coordinates* r *and* t *in terms of* T *and* R *(or in terms of* $\tilde{u}$ *and* $\tilde{v}$*).*

Exercise 2 *Let us consider causal geodesics in the Kruskal extension of the Schwarzschild spacetime.*

1. *What are the possible start and end points of null geodesics for arbitrary initial data? (**Hint:** Discuss the regions* $(I) - (IV)$ *(see fig. 4 in the lecture notes) of the extended Schwarzschild spacetime separately.)*

2. *What can you say about timelike geodesics?*

Exercise 3 *Moving in the Schwarzschild black hole geometry.*

1. *An observer decides to explore the geometry outside a Schwarzschild black hole of mass* M *by starting with an initial velocity at infinity and then falling freely on an orbit that will come close to the black hole and then move out to infinity again. How can the observer arrange to get as close to the black hole as possible on an orbit of this kind and what is the closest (s)he can get to the black hole?*

2. *Once across the event horizon of a black hole, what is the longest proper time the observer can spend before being destroyed in the singularity?*

*(**Hint:** Using the geodesic equations ins Schwarzschild coordinates, show that*

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = e^2 + \left(1 + \frac{\ell^2}{r^2}\right)\left(\frac{2GM}{r} - 1\right) \,, \tag{5}$$

where τ *is the observer's proper time,* e *is the energy per unit mass and* ℓ *is the angular momentum per unit mass.)*