Black Holes and their Thermodynamics[1]

Problem Sheet 4

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Tutorials take place on Mondays, 2-4pm (c.t.) in room A 449, Theresienstr. 37. Please hand in your solutions at the next tutorial on November 11, 2019 or send them in a single pdf in an email before the tutorial.

Exercise 1 *A general, static and spherically symmetric metric in coordinates* (t, r, θ, ϕ) *can be written as*

$$\mathrm{d}s^2 = -A(r)\mathrm{d}t^2 + \frac{1}{B(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \,, \tag{1}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2$ *is the metric on a unit* 2*-sphere. Assume that the* $A(r)$ *and* $B(r)$ *are analytic functions of* r *such that both have a simple zero at* $r = r_+ > 0$ *and are positive for* $r > r_+$.

1. *Show that radial null geodesics are given by* $t \pm r^* = \text{constant}$, *where*

$$r^* = \int_{r_0}^{r} \frac{\mathrm{d}r'}{\sqrt{A(r')B(r')}} \,, \tag{2}$$

   with $r_0 > r_+$ *an arbitrary constant. Show that* $r^* \to -\infty$ *as* $r \to r_+$.

2. *Ingoing Eddington-Finkelstein coordinates are defined by*

$$v = t + r^* \,. \tag{3}$$

   Obtain the metric in these coordinates. Explain why this metric can be analytically continued through $r = r_+$.

3. *For* $A(r) = B(r) = 1 - 2M/r$ *the metric in eq. (1) is the Schwarzschild metric. Which describes a black hole of mass* M. *The Kretschmann invariant is defined by*

$$K = R^{abcd}R_{abcd} \tag{4}$$

   and given by $K = 48M^2/r^6$ *for the spacetime defined by the Schwarzschild metric. Comment on the definitions of spacetime and coordinate singularities.*

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Exercise 2 *In this exercise we will show, that the region $r < 2M$ of the Schwarzschild solution indeed describes a black hole. First, let us introduce the following definitions*

- *A vector is called causal if it is timelike or null. A curve is called causal if its tangent vector is everywhere causal.*
- *Given a time-orientation of the spacetime: a causal vector field T^a. A vector field X^a is called future-directed if it lies in the same lightcone as T^a.*

For the Schwarzschild spacetime let ∂_t denote the time-orientation for $r > 2M$. Show that every future-directed causal curve starting inside the event horizon $r < 2M$ will not cross the event horizon. The following steps may provide a guideline:

1. *Argue that $-\partial_r$ is future-oriented causal.*
2. *Let $x^\mu(\lambda)$ be a future-directed causal curve. Contract the tangent vector $\mathrm{d}x^\mu/\mathrm{d}\lambda$ with $-\partial_r$ to determine the sign of $\dot{v}$.*
3. *After these preliminaries, inspect $\dot{x}^2$ for $r < 2M$.*
4. *In addition, what can you say about future-directed causal curves that start on the event horizon?*

Hint: *Work in ingoing Eddington-Finkelstein coordinates from the previous exercise.*

Exercise 3 *Take the Schwarzschild metric, i.e. set $A(r) = B(r) = 1-2M/r$ in eq. (1).*

1. *Write the Lagrangian of the geodesic equation for the Schwarzschild metric. Derive the components of the geodesic equation. Determine the Christoffel symbols from the geodesic equation.*
2. *Note that the Lagrangian from which geodesics are obtained is independent of t and ϕ. What consequences does this have?*