Black Holes and their Thermodynamics[1]

Problem Sheet 3

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Tutorials take place on Mondays, 2-4pm (c.t.) in room A 449, Theresienstr. 37. Please hand in your solutions at the next tutorial on November 4, 2019 or send them as pdf in an email before the tutorial.

Erratum: Typos in eq. (4) and eq. (13) corrected.

Exercise 1 *Let x^a be a coordinate system and e_a basis vectors. In the lecture we defined the affine connection as*

$$\Gamma^a{}_{bc} = e^a \partial_c e_b \,, \tag{1}$$

from where we found the components of the covariant derivative of the covariant components of a vector field v as

$$\nabla_a v^b = \partial_a v^b + \Gamma^b{}_{ca} v^c \,. \tag{2}$$

1. *By differentiating the reciprocity relation $e^a e_b = \delta^a_b$, find the derivatives of the dual basis vectors e^a with respect to the coordinates, $\partial_b e^a$, in terms of $\Gamma^a{}_{bc}$ from eq. (1).*

2. *Next, take a vector field v and expand it in the dual basis, $v = v_a e^a$, to find the components of the covariant derivative of the contravariant components, $\nabla_a v_b$, using the definition*

$$\partial_a v \equiv (\nabla_a v_b) e^b \tag{3}$$

(analogously to the lecture).

Exercise 2 *Show that a geodesic of spacetime which is timelike (spacelike, null) at one event is timelike (spacelike, null) everywhere.*

Exercise 3 *The Einstein-Hilbert action in 4 dimensions reads*

$$S_{\text{EH}} = \frac{1}{16\pi G} \int \mathrm{d}x^4 \sqrt{-g}(R - 2\Lambda) + \int \mathrm{d}x^4 \sqrt{-g}\mathcal{L}_{\text{matter}} \,, \tag{4}$$

where $g = \det g_{\mu\nu}$ is the determinant of the metric tensor $g_{\mu\nu}$ and R is the Ricci scalar. The matter Lagrangian $\mathcal{L}_{\text{matter}}$ contains all other possible matter fields. The energy-momentum tensor is defined by

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\partial \sqrt{-g}\mathcal{L}_{\text{matter}}}{\partial g^{\mu\nu}} \tag{5}$$

[1] *www.physik.uni−muenchen.de/lehre/vorlesungen/wise_19_20/bh_info_wise_2019_20*

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1. *Calculate the variation of the determinant, $\delta\sqrt{-g}$.*
2. *Show, that the energy momentum tensor can be written as*
$$T_{\mu\nu} = -2\frac{\partial \mathcal{L}_{\text{matter}}}{\partial g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_{\text{matter}} \tag{6}$$
3. *Show by variation that the Einstein-Hilbert action yields the Einstein field equations,*
$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu} - \Lambda g_{\mu\nu}\,. \tag{7}$$
4. *The Lagragian of Maxwell's theory is given by*
$$\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\,, \tag{8}$$
where the field strength tensor $F_{\mu\nu}$ of a vector field A_μ is
$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu\,. \tag{9}$$
Calculate the stress-energy tensor.

*(**Hint:** Express $F_{\mu\nu}$ in terms of partial derivatives first. Does it depend on the metric?)*

Exercise 4 *In the Newtonian limit of weak gravitational fields, the metric is given by*
$$\mathrm{d}s^2 = -(1+2\phi)\mathrm{d}t^2 + (1-2\phi)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)\,, \tag{10}$$
which we encountered before. The energy-momentum tensor of a perfect fluid is given by
$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu - p\, g^{\mu\nu}\,, \tag{11}$$
with ρ the energy density, p the pressure and u^μ the 4-velocity of the perfect fluid.

Assume that the fluid is moving slowly and that the pressure is small, $p \ll \rho$. Show that the 00-component of the Einstein field equations from the previous exercise with a non-zero cosmological constant Λ reduces to
$$\nabla^2\phi = 4\pi G\rho - \Lambda\,, \tag{12}$$
where $\nabla^2 = \delta^{ij}\partial_i\partial_j$. Hence show that the corresponding Newtonian gravitational potential of a spherically symmetric mass M centered at the origin can be written as
$$\phi = -\frac{GM}{r} - \frac{\Lambda r^2}{6}\,. \tag{13}$$
Discuss the difference to the Newtonian gravitational potential.

Plug in the gravitational potential into g_{00} and set $\Lambda = 0$. What can you say about the sign of g_{00} for different values of r?