Black Holes and their Thermodynamics[1]

Problem Sheet 2

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Tutorials take place on Mondays, 2-4pm (c.t.) in room A 449, Theresienstr. 37. Please hand in your solutions at the next tutorial on October 28, 2019.

Erratum: There was a typo in eq. (6), which is corrected in this version.

Exercise 1 *Parallel transport of a vector v along a curve γ is defined by $\nabla_{\dot{\gamma}} v = 0$, where $\dot{\gamma} = \mathrm{d}\gamma/\mathrm{d}\lambda$ and λ is the curve parameter.*

1. *Show explicitly that parallel-transport preserves the norm of a vector.*

2. *Set up the equations for parallel transport of a vector v along a curve γ on the two-sphere S^2 (cf. Exercise 2 of the first problem sheet)*

3. *Solve these equations generally for transport along meridians ($\phi = \mathrm{const}$) or latitude curves ($\theta = \mathrm{const}$).*

4. *Shift a unit vector originally pointing north along the following curve: north along a meridian from the equator to $\theta = \pi/4$, then east along a latitude circle by $\Delta\phi = \pi/4$, then south along a meridian back to the equator, then west along the equator to where it started. Where is the vector pointing now?*

5. *Repeat the calculations for the same vector on the cylinder described by the metric $\mathrm{d}s^2 = \mathrm{d}z^2 + \mathrm{d}\phi^2$. The vector should be transported along the following closed path: start from the position $(\phi, z) = (0, 0)$ and move east for $\Delta\phi = \pi/4$, then north along a straight line perpendicular to the plane $z = 0$ for $\Delta z = \sqrt{2}/2$, then west for $\Delta\phi = \pi/4$ along an arc of circle parallel to the plane $z = 0$ and then south to the starting point. Where does the vector point now?*

6. *What do you conclude from these results?*

Exercise 2 *The Weyl tensor in D spacetime dimensions is defined as*

$$C^{\mu}{}_{\nu\rho\sigma} = R^{\mu}{}_{\nu\rho\sigma} + \frac{2}{D-2}\left(\delta^{\mu}_{[\sigma}R_{\rho]\nu} - g_{\nu[\sigma}R_{\rho]}{}^{\mu}\right) + \frac{2}{(D-1)(D-2)}R\delta^{\mu}_{[\rho}g_{\sigma]\nu}\,. \quad (1)$$

[1] $www.physik.uni-muenchen.de/lehre/vorlesungen/wise\_19\_20/bh\_info\_wise\_2019\_20$

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Show that it has the same symmetries like the Riemann tensor and also that it is trace-free:

1. $C_{\mu\nu\rho\sigma} = -C_{\mu\nu\sigma\rho}$
2. $C_{\mu\nu\rho\sigma} = C_{\rho\sigma\mu\nu}$
3. $C^{\mu}{}_{[\nu\rho\sigma]} = 0$
4. $C^{\mu}{}_{\nu\mu\sigma} = 0.$

Exercise 3 *Conformal transformations change the metric by a scalar factor*

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = e^{2\Lambda} g_{\mu\nu} \,, \tag{2}$$

where $\Lambda = \Lambda(x^\mu)$ may depend on all coordinates x^μ. In this exercise we will show that conformal transformations leave the Weyl tensor from the previous exercise invariant.

1. *Show that the Christoffel symbols transform as*

$$\Gamma^{\mu}_{\nu\rho} \to \tilde{\Gamma}^{\nu}_{\nu\rho} = \Gamma^{\nu}_{\nu\rho} + S^{\mu}_{\nu\rho} \tag{3}$$

where

$$S^{\mu}_{\nu\rho} = \delta^{\mu}_{\nu} \nabla_\rho \Lambda + \delta^{\mu}_{\rho} \nabla_\nu \Lambda - g_{\nu\rho} \nabla^\mu \Lambda \tag{4}$$

2. *Next, calculate the Riemann tensor to show*

$$\tilde{R}^{\mu}_{\nu\rho\sigma} = R^{\mu}_{\nu\rho\sigma} + \left(\nabla_\rho S^{\mu}_{\sigma\nu} + S^{\mu}_{\delta\rho} S^{\delta}_{\sigma\nu} - \rho \leftrightarrow \sigma \right) \,, \tag{5}$$

where $-\rho \leftrightarrow \sigma$ means subtract the two terms in the bracket obtained by interchanging the ρ and σ indices. By writing out the two terms in the brackets, and identifying terms that are symmetric in the ρ, σ indices, show that one can eventually write

$$\begin{aligned} \tilde{R}^{\mu}_{\nu\rho\sigma} = R^{\mu}_{\nu\rho\sigma} &+ 2\delta^{\mu}_{[\sigma} \nabla_{\rho]} \nabla_\nu \Lambda - 2 g_{\nu[\sigma} \nabla_{\rho]} \nabla^\mu \Lambda + 2\delta^{\mu}_{[\rho} \nabla_{\sigma]} \Lambda \, \nabla_\nu \Lambda \\ &- 2\delta^{\mu}_{[\rho} g_{\sigma]\nu} (\nabla \Lambda)^2 - 2 g_{\nu[\rho} \nabla_{\sigma]} \Lambda \nabla^\mu \Lambda \,. \end{aligned} \tag{6}$$

3. *By contracting eq. (6), calculate the Ricci tensor and Ricci curvature of $\tilde{g}$ in D spacetime dimensions.*

4. *Finally show that the Weyl tensor is invariant under conformal transformations, $\tilde{C}^{\mu}{}_{\nu\rho\sigma} = C^{\mu}{}_{\nu\rho\sigma}$.*

Thus a conformally flat metric of the form $\mathrm{d}\tilde{s}^2 = e^{2\Lambda} \mathrm{d}s^2$ (flat) has a vanishing Weyl tensor (the converse is also true for $D \geq 4$).